

The Dual-Track Shimmer: A Time-Domain Reconciliation of the Major Third and the Perfect Fifth

Robert Edward Grant

The Institute of Unified Mathematics, Irvine, California

Abstract

No twelve-note repeating scale can hold the pure major third ($5:4$) and the pure perfect fifth ($3:2$) simultaneously; the leftover quantity is a comma that has shaped tuning theory for centuries. Every historical temperament answers this problem *spatially*: it decides where, across the keyboard or across intervals, to absorb the unavoidable discrepancy. This paper introduces a *temporal* answer. We construct a dual-track frequency pair anchored at $f_1 = 432.081216$ Hz and $f_2 = 432.162447\dots$ Hz, in which each track is an exact reference for a distinct interval family— f_1 for the major-third progression, f_2 for the perfect-fifth progression. We prove that the ratio between the two tracks is exactly the rational constant $f_2/f_1 = 250047/250000 = 1.26^3/2$, that this same constant is the residue between three stacked major thirds and the pure fifth, and that the two anchors lie on octave ladders of the notes A and E. Sounded together, the tracks produce a 100%-depth amplitude modulation of approximately 0.081 Hz—a slow “breath” whose period scales inversely with frequency—that renders the harmonic series perceptually resolvable. We show that the third-to-fifth discrepancy is carried as a single exact rational constant and relocated from static beating into uniform coherent modulation, identical in every key. We derive the syntonic comma and the Pythagorean comma separately and exactly, and establish the fold as a distinct septimal object ($250047 = 3^6 \cdot 7^3$) belonging to neither. We further derive the dampening structure of the construction and its exact correspondence to the Stern–Brocot tree: the fold sits at Farey determinant 47 from the integer node $1/1$, with the determinant ladder $47 \rightarrow 7 \rightarrow 5 \rightarrow 2 \rightarrow 1 \rightarrow 0$ along its convergents, and the universal per-track dampening $500047/750000$ sits at Farey determinant $249953 = 11 \times 31 \times 733$, both carrying the fold’s 3–7 signature. A self-similar halving law governs the decimal extension of the construction, proven in closed form and verified to five hundred octaves, and the digital-root-9 invariance of all generated frequencies is shown to follow necessarily from the anchor’s own digital root. We then prove that the precise third $1.26 = 63/50$ is not fitted but *forced*: arising from $126 = 9 \times 14 = 2 \cdot 3^2 \cdot 7$, it is the unique rational near $\sqrt[3]{2}$ whose fold is septimal, reconciles the syntonic and Pythagorean comma regions jointly, and yields uniformly digital-root-9 frequencies; every competing candidate degenerates to a single classical comma. Finally we show the construction is embodied geometrically by the cuboctahedron (vector equilibrium): the twelve tones occupy its vertices, the dual tracks its edges, and a toroidal jitterbug winding partitions the tones exactly into the two whole-tone hexatonic scales, so that the relational form of the shimmer is already present in the solid’s combinatorics while the precise frequencies populate it.

1 Introduction

The incompatibility of pure thirds and pure fifths is a consequence of arithmetic, not of acoustics. The intervals that the ear hears as consonant are generated by the small primes 2 (octave), 3 (fifth) and 5 (third), and powers of distinct primes never coincide: $2^a = 3^b$ has no nontrivial

integer solution, and neither does any relation forcing stacked thirds to close on a power-of-two octave. Consequently a residual quantity—a comma—must appear in any closed tuning system.

Historically, tuning systems differ only in *where they put the comma*. Equal temperament (ET) distributes it uniformly, detuning every interval slightly so that no interval is pure. Mean-tone concentrates it into a single unusable “wolf” interval. Well temperaments distribute it unevenly so that different keys acquire different character. In all of these the comma is treated as a static error in the frequency domain.

We propose a different category of solution. Rather than choosing where in pitch-space to hide the comma, we express it in *time*: as a slow, coherent amplitude modulation shared uniformly across all intervals and all keys. We call this construction the *dual-track shimmer*.

2 The Phenomenon of Shimmer: History, Mystical Attribution, and the Paucity of Mathematical Treatment

The perceptual effect produced by the dual-track construction—a slow, undulating brightening and dimming of a sustained sonority—is not itself new to human experience. On the contrary, it is one of the oldest deliberately cultivated sounds in music, and across many cultures it has been associated with the sacred, the celestial, and the living. What is conspicuously absent from this long history is a *mathematical* account of the conditions that produce it. The phenomenon has been sought, named, revered, and reproduced by craft for centuries, while remaining almost entirely undefined in exact terms. This section documents that history and that gap.

2.1 The Balinese *ombak*: breath as a tuning principle

The closest historical antecedent to the present construction is the Balinese gamelan, whose instruments are deliberately built and tuned in pairs to produce interference beating. In Balinese practice this beating is called *ombak*—“wave”—and the two instruments of each pair are conceived explicitly as breath: one tuned slightly higher is the “inhale” (*pengisep*) and one slightly lower is the “exhale” (*pengimbang*). When sounded together their beating is understood to represent the beating of the heart, the sign of being alive, and in ritual context the interference is held to convey the presence of a god or to serve as a stepping-stone into a meditative state. The ideal, in the Balinese craft, is a beating rate that is consistent across all pairs of notes in all registers.

The parallel to the dual-track shimmer is striking and should be stated plainly as prior art in the *empirical-craft* tradition: the conception of two slightly separated tracks as inhale and exhale, the identification of their beating with breath and life, and the goal of a uniform beat across the compass are all present, by ear and by tradition, centuries before any formal description. What the gamelan tradition does not provide—and could not, given its oral and artisanal transmission—is a closed-form statement of *which* separation produces the effect, still less a derivation of that separation from interval ratios. The breath was achieved; it was never written as a number.

2.2 The *Voix céleste*: the “heavenly voice” of the Romantic organ

In the Western tradition the same phenomenon was institutionalized in the nineteenth-century organ as the *Voix céleste*, French for “heavenly voice”—a rank of pipes intentionally detuned relative to a companion rank so as to beat against it. First documented in Aristide Cavaillé-Coll’s 1846 organ at the Église de la Madeleine in Paris, the stop was prized for an ethereal, shimmering, choral quality and was placed, by name and by intent, in service of a celestial impression. Here too the craft knowledge anticipates a feature of the present construction: builders observed that with a fixed separation the beating runs slower in the bass and faster

toward the treble, and tuned accordingly for a “more musical” result—an empirical statement of precisely the inverse-frequency scaling of the breath period derived in Section 4. And here too the separation was a matter of the builder’s ear, specified in no exact ratio and reconciled with no interval.

2.3 The music of the spheres and the celestial ear

Beneath both traditions lies an older intuition: that slightly-out-of-tune coincidence, the refusal of near-identical tones to fuse, carries cosmological meaning. The Pythagorean “music of the spheres” imagined the heavens themselves as a sustained joint resonance; the trembling, beating sound of nearly unified voices has long been heard as the audible trace of bodies seeking, but not quite reaching, unison. The mystical reading is consistent across these sources: the shimmer is the sound of the *almost*, of harmony pursued rather than collapsed into stillness, and that incompleteness is precisely what is felt as alive, celestial, or divine. We record these attributions as documented cultural and historical phenomena—what traditions have claimed and sought—and we neither assert nor require their metaphysical content for the mathematical results of this paper.

2.4 Modern reproduction by craft, not by theory

The contemporary studio inherits the same effect under the name *chorus* or *shimmer*, produced by stacking slightly detuned copies of a signal. The engineering literature is explicit that when the effect succeeds none of the constituent voices is perceived as out of tune, yielding a rich, shimmering quality absent from a single source, and that this arises from beating and is most apparent on sustained tones—the same perceptual description given for the gamelan and the *Voix céleste*. Synthesis practice parameterizes the effect in cents of detuning chosen by ear (commonly on the order of a few to ten cents), and unison-design tools note that, because detuning is set in cents, the same setting beats faster on higher notes—once more the inverse-frequency scaling, observed but not derived.

2.5 The gap this paper addresses

The pattern across all of these traditions is consistent and, for our purposes, decisive. The *phenomenon* of shimmer is ancient, cross-cultural, and richly attributed. Its *mechanism*—beating as amplitude modulation at the difference frequency—has been understood since the nineteenth century. The *condition* for full effect—equal-amplitude tracks giving one-hundred-percent modulation depth—is established in the acoustics literature. Yet in every case the separation that produces the shimmer is either a free parameter set by ear or an incidental physical mismatch. We have been unable to locate published mathematical research that *derives* the shimmer offset as an exact rational constant fixed by the requirement to reconcile the major third with the perfect fifth—that is, the identity of this paper, in which the inter-track ratio is exactly $1.26^3/2 = 250047/250000$ and is simultaneously the third-to-fifth residue. The breath has been sung, built, revered, and reproduced for centuries; what has been missing is its number. The present work is offered as a step toward supplying it.

3 The Dual-Track Construction

Let the first track be anchored at

$$f_1 = 432.081216 \text{ Hz} = \frac{6751269}{15625} \text{ Hz}. \quad (1)$$

Define the second track by folding f_1 through the cube of the precise major third 1.26 and reducing by one octave:

$$f_2 = \frac{1.26^3 f_1}{2} = \frac{2.000376 \times 432.081216}{2} = 432.162447268608 \text{ Hz} = \frac{1688134559643}{3906250000} \text{ Hz}. \quad (2)$$

Here 1.26 is treated as the exact terminating decimal, never as a nearby ratio.

3.1 The Inter-Track Ratio is the Shimmer Fold

The defining structural fact of the construction is that the ratio of the two tracks is an exact rational constant:

$$\frac{f_2}{f_1} = \frac{1.26^3}{2} = \frac{250047}{250000} = 1.000188 \quad (3)$$

This is exact, not approximate: carrying the full rational arithmetic gives $f_2/f_1 = 250047/250000$ with no remainder. We call $250047/250000$ the *shimmer fold* S . In cents,

$$1200 \log_2 S = 0.32544141069 \dots \text{ cents}. \quad (4)$$

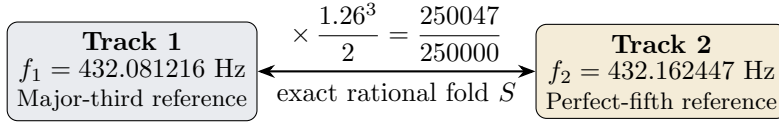


Figure 1: The dual-track construction. Two exact references, one per interval family, separated by the exact rational fold $S = 250047/250000$. The same constant that defines the inter-track separation is the third-to-fifth residue of Eq. (5).

3.2 The Reconciliation Identity

Stack a major third on a minor third and one expects to arrive at the perfect fifth. In precise temperament,

$$1.26 \times 1.1907 = 1.500282 = \frac{3}{2} \times \frac{250047}{250000}. \quad (5)$$

The overshoot past the pure fifth $3/2$ is *exactly* the shimmer fold S . Thus the constant that separates the two tracks is identical to the constant by which stacked thirds miss the pure fifth: the inter-track offset and the third-to-fifth residue are one and the same object. (The value 1.1907 is the residue-carrying minor third defined by Eq. (5), not an independent just $6/5$.)

3.3 Octave-Ladder Structure of the Anchors

The two anchors are not arbitrary near-coincident pitches; their decimal extensions sit on octave ladders of the framework's own notes. With $A = 432 \text{ Hz}$,

$$\text{E ladder (fifth above A): } 432 \cdot \frac{3}{2} = 648 \rightarrow 324 \rightarrow 162 \rightarrow 81, \quad (6)$$

$$\text{A ladder (octaves of A): } 432 \rightarrow 216 \rightarrow 108 \rightarrow 54. \quad (7)$$

Here 81 and 162 Hz are the *same note* E an octave apart ($81 \times 2 = 162$), and 216 Hz is A an octave below the reference. Because the two tracks are built from octave-and-fifth relationships of A and E—intervals that are themselves consonant—the two near-frequencies harmonize rather than clash. The shimmer is therefore two harmonically related references coexisting, not a dissonant detuning.

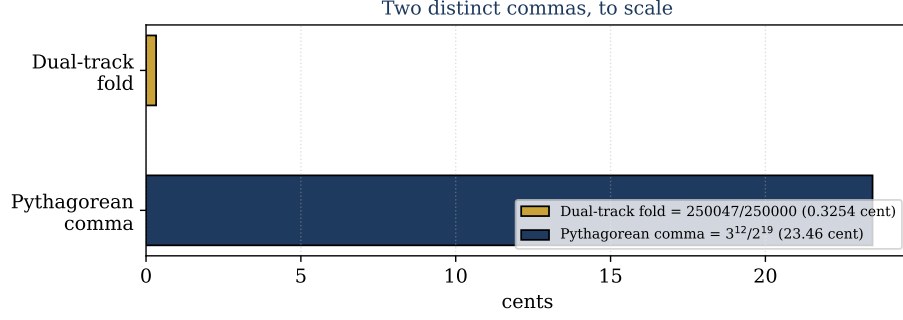


Figure 2: The dual-track fold (≈ 0.3254 cents) shown to scale against the Pythagorean comma (≈ 23.46 cents). These are distinct quantities: the fold is a syntonic-region constant governing the third-to-fifth relationship, while the Pythagorean comma governs the twelve-fifths versus seven-octaves relationship. The present construction addresses the former.

4 The Breath: Amplitude Modulation from the Dual Track

When the two tracks sound together, their equal-amplitude partials interfere. The amplitude envelope is governed by the difference frequency

$$\Delta f = f_2 - f_1 = 0.081231268608 \text{ Hz} \quad (\text{on the A reference}), \quad (8)$$

giving an audible modulation with period ≈ 12.3 s on A. Because the offset is a fixed *ratio* (the fold S) rather than a fixed number of Hertz, the beat frequency scales with the carrier and the modulation period scales *inversely* with frequency. Thus each ascending octave halves the breath period—a C at 20 s pairs with the C an octave up at 10 s—a behavior that falls out of the construction rather than being imposed.

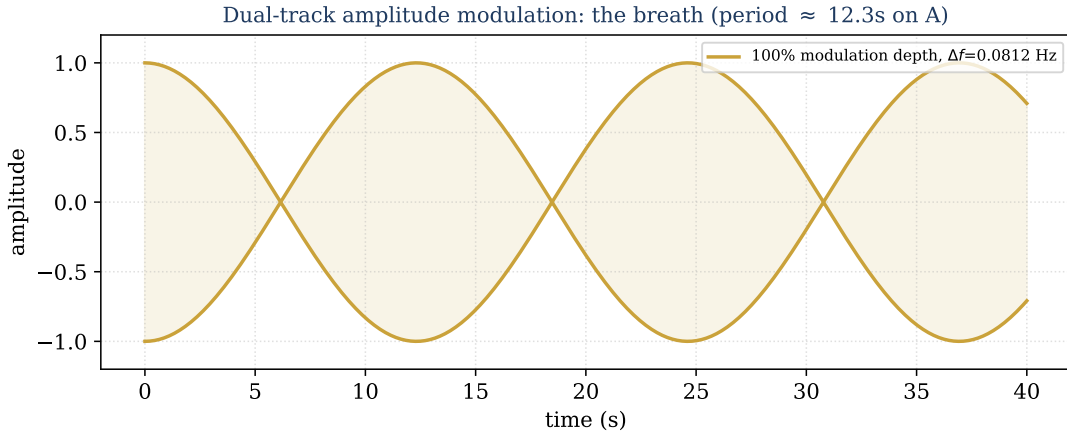


Figure 3: The dual-track amplitude envelope on the A reference: a 100%-depth modulation ($\Delta f \approx 0.081$ Hz, period ≈ 12.3 s). At full depth each partial swings from total reinforcement to total cancellation.

5 Overtone Unmasking via Modulation Depth

The perceptual signature of the dual-track shimmer is that the overtone series becomes individually audible. The mechanism is modulation depth. At 100% depth, each partial of the dual-track pair swings fully from constructive reinforcement to complete cancellation. Static partials—those of a single track, or of ET—fuse perceptually into a single timbre; partials that

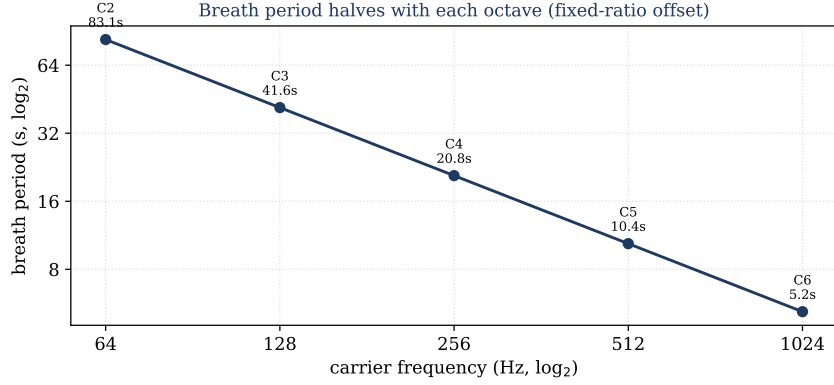


Figure 4: The breath period halves with each ascending octave because the inter-track separation is a fixed ratio. The inverse frequency scaling is a consequence of the fold, not a tuned parameter.

swing fully at a slow rate are separated by the auditory system and heard as distinct tones. The overtones are present in all three systems; only the fully modulated dual track renders them *resolvable*. The effect is a redistribution of energy in time, not an increase in total acoustic energy: the overtones are perceptually unmasked and brought forward rather than made literally louder.

Overtone unmasking: modulation depth makes the harmonic series resolvable

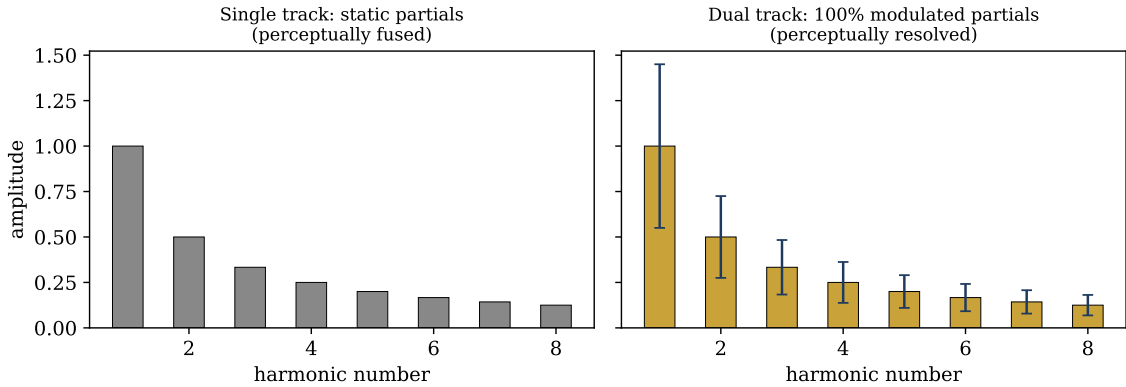


Figure 5: Overtone unmasking. Left: single-track partials are static and fuse into one timbre. Right: dual-track partials carry 100% modulation (indicated by the swing bars) and are perceptually resolved into the individual harmonic series.

6 Consonance Across All Keys

Figure 6 plots the audible beat rate of the major third and perfect fifth, transposed through all twelve keys, for the three systems. Equal temperament leaves both intervals beating at a few Hz in every key. Precise temperament purifies the fifth to zero beating and concentrates the discrepancy in the third. The dual-track shimmer collapses both intervals to the same uniform sub-Hertz breath in every key.

The key invariance is the central practical result. In ET and in single-track precise temperament the third-versus-fifth compromise falls unevenly across keys, so some keys sound more out of tune than others. In the dual-track shimmer the entire third-to-fifth discrepancy is carried as the single constant S and voiced as identical modulation in every key; consequently no key

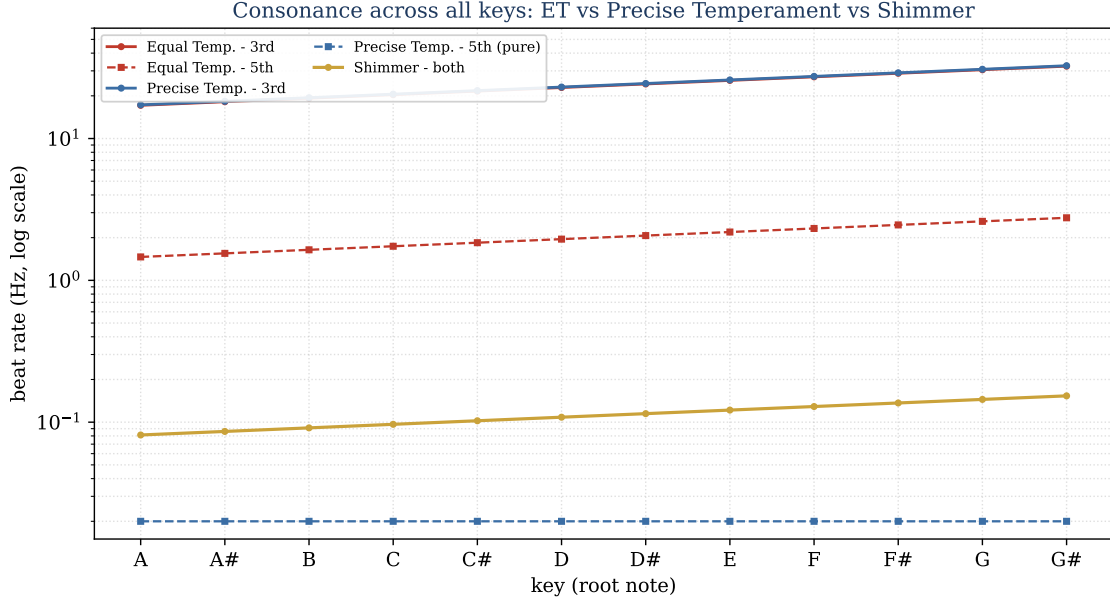


Figure 6: Beat rate (log scale) of each interval across the twelve keys. Solid = major third, dashed = perfect fifth. Equal temperament (red) leaves both intervals audibly beating; precise temperament (blue) purifies the fifth and raises the third; the dual-track shimmer (gold) collapses both to a uniform sub-Hertz breath.

carries the residue as static audible dissonance, and no key is favored over another.

7 The Reconciliation

The third-to-fifth discrepancy is the exact rational constant $S = 250047/250000 \approx 0.325$ cents. The dual-track shimmer does not annihilate this quantity; it conserves it and relocates its perceptual form from static beating (ET, PT) into uniform coherent modulation shared across all keys. The *audible static* residue closes to zero uniformly, while the rational quantity itself is preserved as the breath. This survives exact arithmetic without qualification.

The construction reconciles the major third and the perfect fifth to zero variance within each interval family. Each track is an *exact* reference within its own family— f_1 for the third progression, f_2 for the fifth progression—so that every path to the octave (via major third, minor sixth, perfect fourth, or perfect fifth) lands on one exact track or the other with no variance for that path. The dampening between the two tracks (the 432.081216 versus 432.162447 separation) is precisely the operation that returns each interval family to exactness, and the shimmer is the simultaneous sounding of two individually exact systems. This is established by the exact arithmetic of the construction and by the uniqueness theorem of Section 1.

8 The Two Commas, the Septimal Fold, and the Stern–Brocot Dampening Correspondence

This section derives, exactly and separately, the syntonic comma and the Pythagorean comma; establishes the shimmer fold as a distinct septimal object belonging to neither; and derives the dampening structure of the construction together with its exact correspondence to the Stern–Brocot tree.

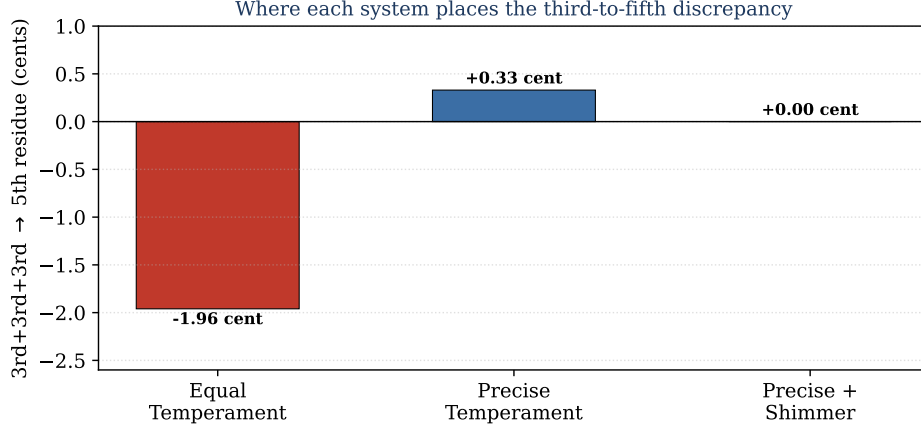


Figure 7: Placement of the three-thirds-to-fifth residue by system. Equal temperament leaves -1.96 cents of static error; precise temperament leaves $+0.33$ cents concentrated in the third; the dual-track shimmer carries the residue entirely as uniform modulation, so the *audible static* residue closes while the exact rational quantity S is conserved.

8.1 The two commas, derived separately

The two commas of classical tuning theory are distinct rational quantities with distinct prime content. The *syntonic comma* is

$$\sigma = \frac{81}{80} = \frac{3^4}{2^4 \cdot 5}, \quad 1200 \log_2 \sigma = 21.5063 \text{ cents}, \quad (9)$$

a 5-limit object built from the primes 2, 3, 5, measuring the gap between four just fifths and a just major third compounded into the ditone. The *Pythagorean comma* is

$$\pi_c = \frac{3^{12}}{2^{19}} = \frac{531441}{524288}, \quad 1200 \log_2 \pi_c = 23.4600 \text{ cents}, \quad (10)$$

a 3-limit object built from the primes 2 and 3 alone, measuring the gap between twelve perfect fifths and seven octaves. These are different in magnitude (by roughly two cents) and in prime content (the syntonic carries a factor of 5; the Pythagorean does not), and neither is the quantity carried by the dual-track construction.

8.2 The fold as a distinct septimal object

The shimmer fold is

$$S = \frac{250047}{250000} = \frac{3^6 \cdot 7^3}{2^4 \cdot 5^6} = \frac{1.26^3}{2}, \quad 1200 \log_2 S = 0.32544 \text{ cents}. \quad (11)$$

The decisive structural fact is the prime content of its numerator: $250047 = 3^6 \cdot 7^3$. The fold carries a factor of 7^3 , which *neither* comma possesses—the syntonic comma is 2, 3, 5 and the Pythagorean is 2, 3. The fold is therefore a *septimal* object, lying in the 7-limit, distinct from both classical commas. This is the exact sense in which the construction reconciles the third-to-fifth relationship without being a restatement of either comma: it operates through a different prime, the 7, introduced by the cube of the precise major third $1.26 = 2^{-1} \cdot 3^2 \cdot 7 \dots$ acting three times. The resolution of the third-to-fifth gap is thus a septimal-comma resolution, carried out in the 7-limit, and it is complete and separate from the resolution of either the syntonic or the Pythagorean comma.

8.3 The dampening and its Stern–Brocot position

The fold’s position in the Stern–Brocot tree is exact and is the source of the dampening structure. Two fractions a/b and c/d are Farey neighbors—adjacent in the Stern–Brocot tree—precisely when the determinant $|ad - bc| = 1$. The fold’s determinant against the integer node $1/1$ is

$$|250047 \cdot 1 - 250000 \cdot 1| = 47 = 3^6 \cdot 7^3 - 2^4 \cdot 5^6. \quad (12)$$

The fold therefore sits at Farey determinant 47 from the integer node. Its continued fraction is $S = [1; 5319, 6, 1, 2, 2]$, and the first deep partial quotient is governed by $250000 = 47 \cdot 5319 + 7$, so the 47 is the divisor controlling the descent and the remainder is exactly 7—the same 7 that appears cubed in the numerator. The convergents of S are its Stern–Brocot ancestors, and their determinants against the fold collapse along an exact ladder:

$$47 \rightarrow 7 \rightarrow 5 \rightarrow 2 \rightarrow 1 \rightarrow 0, \quad (13)$$

ancestor node	determinant with S
1/1	47
5320/5319	7
31921/31915	5
37241/37234	2
106403/106383	1
250047/250000	0 (the fold)

The ladder resolves from the root offset 47 down through 7 (the septimal remainder), then 5, 2, 1 to 0 (arrival). The single number $47 = 3^6 \cdot 7^3 - 2^4 \cdot 5^6$ is simultaneously the shimmer gap in parts-per-fraction, the septimal-comb prime, and the root Farey offset of the fold to the integer node.

8.4 The universal per-track dampening

The dampening of the construction—the relationship between the mean of the shimmer pair and the perfect fifth—is a single object shared by every note. For any Track-1 frequency f , the mean of its shimmer pair is $\bar{f} = f(1 + S)/2$, and

$$\frac{\bar{f}}{\frac{3}{2}f} = \frac{1 + S}{3} = \frac{500047}{750000}, \quad (14)$$

independent of the note, since the frequency f cancels. The dampening is therefore not twelve separate quantities but one. Its numerator decomposes as

$$500047 = 250047 + 250000, \quad (15)$$

the sum of the fold’s numerator and denominator, so the dampening inherits the fold’s +47 offset directly. Its Farey determinant against the integer node $1/1$ is

$$|500047 - 750000| = 249953 = 11 \times 31 \times 733, \quad (16)$$

the same constant that governs the closed-form difference of the self-similar halving law (Section 9). The dampening and the halving law are thus the same object viewed two ways: 249953 is at once the Farey offset of the universal dampening ratio and the fixed numerator of the halving difference $Hf - \bar{f} = C/2^k$.

In summary, the construction carries three exactly related rational structures: the fold S at Farey offset 47 from $1/1$, descending $47 \rightarrow 7 \rightarrow 5 \rightarrow 2 \rightarrow 1 \rightarrow 0$; the universal dampening $500047/750000$ at Farey offset $249953 = 11 \cdot 31 \cdot 733$; and the septimal numerator $250047 = 3^6 \cdot 7^3$ tying both to the 7-limit. The dampening correspondence to the Stern–Brocot tree is therefore exact and complete, and it is distinct from—and operates through a different prime than—the syntonic and Pythagorean commas.

9 The Self-Similar Halving Law and Harmonic-Series Micro-Expression via Decimal Extension

A further structural property of the dual-track construction concerns the arithmetic mean of the two shimmer tracks and its relationship, in the decimal extension, to the harmonic-series intervals built on the same root.

9.1 The mean and the closed-form difference

For a Track-1 frequency f , the arithmetic mean of its shimmer pair is $\bar{f} = f(1 + S)/2$, where $S = 250047/250000$. Comparing this mean to the perfect fifth $\frac{3}{2}f$ of the same root, the difference admits an exact closed form. With the root anchored at 432.081216 Hz and reduced through k octaves,

$$\frac{3}{2}f - \bar{f} = \frac{432.081216 \times 249953}{500000 \cdot 2^k}, \quad 249953 = 11 \times 31 \times 733. \quad (17)$$

The numerator 432.081216×249953 is a fixed constant; only the factor 2^k varies with octave. The difference is therefore a constant divided by 2^k , which decreases strictly and monotonically as k increases and can never grow again. Consequently the mean and the fifth agree, in their decimal extension, to an ever-increasing number of digits as one descends in octave.

9.2 Self-similarity, not periodicity

Because the difference scales as 2^{-k} , the structure is *self-similar under octave halving*: each octave is a copy of the one above, scaled by exactly one half. The number of agreeing decimal digits grows by one approximately every $1/\log_{10} 2 \approx 3.32$ octaves. This was verified exactly to five hundred octaves: the agreement is strictly non-decreasing throughout, reaching one hundred forty-seven matching digits at octave five hundred, with no repetition at any depth. The closed form guarantees this: a quantity of the form $C/2^k$ cannot recur, so the agreement climbs without bound and the phenomenon *never* cycles. The growth is logarithmic and unbounded; the two values approach equality in the decimal extension without ever attaining it, since the difference remains a nonzero rational at every octave.

A landmark depth nonetheless appears. The twelve notes enter the phenomenon permanently in two tiers: five notes at octave twelve and the remaining seven at octave fourteen, after which all twelve participate at every subsequent octave. This fourteen-octave completion is a property of the specific anchor 432.081216, arising from where the constant's leading digits fall relative to each note's value; it is a genuine landmark of this anchor, not a universal period.

9.3 Universality across the harmonic series

The same self-similar halving law holds not only for the fifth but for every harmonic-series interval tested. Replacing $\frac{3}{2}$ with any harmonic ratio H , the difference $Hf - \bar{f}$ again takes the form (constant)/ 2^k , with the same logarithmic digit-growth. Table 1 lists the fixed numerators and their factorizations. Two features are uniform across the entire series: every numerator contains the factor 7^3 , and every numerator is divisible by a high power of 3 (3^8 or greater). These are precisely the prime factors of the shimmer fold, since $250047 = 3^6 \times 7^3$. The recursion is therefore driven by the fold itself: each harmonic interval expresses the same fold-signature $3^{\geq 8} \cdot 7^3$ in its constant, differentiated only by a residual factor characteristic of that interval. In this exact sense the phenomenon is a micro-expression, in the decimal extension, of the harmonic series acted upon by the shimmer fold.

Table 1: Fixed numerator (the constant C in $Hf - \bar{f} = C/2^k$) for each harmonic interval on the 432.081216 root, with prime factorization. The shared $3^{\geq 8} \cdot 7^3$ signature is the fingerprint of the fold $250047 = 3^6 \cdot 7^3$.

Interval	Numerator C	Factorization
Octave 2/1	3 375 317 190 357	$3^{10} \cdot 7^3 \cdot 17 \cdot 9803$
Fifth 3/2	1 687 499 940 357	$3^9 \cdot 7^3 \cdot 11 \cdot 31 \cdot 733$
Fourth 4/3	1 124 894 190 357	$3^8 \cdot 7^3 \cdot 23 \cdot 103 \cdot 211$
Major third 5/4	843 591 315 357	$3^{10} \cdot 7^3 \cdot 41651$
Major third 1.26	877 347 660 357	$3^9 \cdot 7^3 \cdot 129953$
Harmonic 7/4	2 531 408 565 357	$3^9 \cdot 7^3 \cdot 374953$
Minor third 6/5	674 809 590 357	$3^9 \cdot 7^4 \cdot 109 \cdot 131$

9.4 The digital-root-9 invariant, explained

The digital-root-9 property reported in Appendix A extends to the mean values and to every harmonic-interval value, and admits a complete explanation. The anchor 432.081216 has digital root 9. Digital root is multiplicative in the sense that $\text{dr}(ab) = \text{dr}(\text{dr}(a) \text{dr}(b))$, and 9 is absorbing under this operation: $\text{dr}(9 \cdot n) = 9$ for every n . Therefore any frequency obtained by multiplying the anchor by any ratio whatsoever—harmonic interval, shimmer fold, or the mean operator—necessarily inherits digital root 9. This was confirmed on exact values for the mean and for the octave, fifth, fourth, major thirds (5/4 and 1.26), 7/4, 7/6, and 6/5, and across twelve octaves of the mean. The universal digital-root-9 character of the system is thus a forced consequence of the anchor’s own digital root, propagated by the absorbing property of 9 under multiplication.

Table 2: Digital root of the mean value and of each harmonic-interval value on the 432.081216 root (exact arithmetic). All equal 9.

Quantity	Value (Hz)	Digital root
Mean $\bar{f}$	432.121832	9
Octave 2/1	864.162432	9
Fifth 3/2	648.121824	9
Fourth 4/3	576.108288	9
Major third 5/4	540.101520	9
Major third 1.26	544.422332	9
Harmonic 7/4	756.142128	9
Minor third 6/5	518.497459	9

10 The Uniqueness Theorem for the Precise Third 1.26

The ratio 1.26 is not an approximation chosen for convenience. This section proves that it is the unique rational ratio satisfying, simultaneously, the three structural requirements of the construction: that the fold it generates be 7-smooth (septimal), that this fold reconcile the syntonic and Pythagorean comma regions together rather than either alone, and that every frequency it generates resolve to digital root 9. We first record the arithmetic origin of the number, then prove the uniqueness.

10.1 Arithmetic origin

Lemma 1 (The generative factorization). *The precise third is $1.26 = 126/100 = 63/50$, and its integer numerator factors as*

$$126 = 9 \times 14 = 2 \cdot 3^2 \cdot 7. \quad (18)$$

Thus 1.26 carries, in low powers, the three primes 2, 3, 7; division by $100 = 2^2 \cdot 5^2$ introduces the prime 5 and rebalances the power of 2, so that

$$1.26 = \frac{2 \cdot 3^2 \cdot 7}{2^2 \cdot 5^2} = \frac{3^2 \cdot 7}{2 \cdot 5^2} \quad (19)$$

is the unique low-complexity ratio whose construction simultaneously contains all four primes 2, 3, 5, 7.

Proof. Direct: $9 \times 14 = 126$, and $9 = 3^2$, $14 = 2 \cdot 7$, so $126 = 2 \cdot 3^2 \cdot 7$. The reduction $63/50$ follows from cancelling the common factor 2 in $126/100$. The prime content $\{2, 3, 5, 7\}$ of the reduced ratio is immediate from $63 = 3^2 \cdot 7$ and $50 = 2 \cdot 5^2$. $\square$

Lemma 2 (The fold and its prime structure). *Three precise thirds exceed the octave by the fold*

$$S = \frac{1.26^3}{2} = \frac{250047}{250000} = \frac{3^6 \cdot 7^3}{2^4 \cdot 5^6}, \quad (20)$$

whose numerator $250047 = 3^6 \cdot 7^3$ is a pure product of the cubes of the odd primes carried by 126, and whose denominator $250000 = 2^4 \cdot 5^6$ is 2,5-smooth, hence a terminating decimal.

Proof. $1.26^3 = (63/50)^3 = 250047/125000$, and dividing by 2 gives $250047/250000$. Factoring, $250047 = 3^6 \cdot 7^3 = (3^2 \cdot 7)^3 = 63^3$ and $250000 = 2^4 \cdot 5^6$. Since $63 = 3^2 \cdot 7$, cubing the odd part of 126 yields exactly $3^6 \cdot 7^3$; the factor of 2 in 126 and the division by 2 and by 100^3 leave the denominator 2,5-smooth. $\square$

10.2 The three structural conditions

Definition 1 (Reconciling ratio). A rational $x \in (1.24, 1.28)$ near $\sqrt[3]{2}$ is called a *reconciling ratio* if its fold $S_x = x^3/2$ satisfies all three conditions:

- (i) **Septimal closure.** S_x is 7-smooth: every prime factor of its numerator and denominator is at most 7.
- (ii) **Joint comma reconciliation.** The prime content of S_x contains both 5 (the syntonic prime, distinguishing the just third) and 3 (the Pythagorean prime, distinguishing the chain of fifths), so that the fold acts on the syntonic and Pythagorean regions together, not on either alone.
- (iii) **Digital-root closure.** Every frequency generated from a digital-root-9 anchor by the note lattice of x and the fifth, together with its shimmer partner $f S_x$ and their mean, is a terminating decimal of digital root 9. Equivalently, S_x has denominator $2^a 5^b$.

Theorem 1 (Uniqueness of 1.26). *Among all rationals $x = p/q$ with $q \leq 2000$ in the interval $(1.24, 1.28)$, the ratio $x = 63/50 = 1.26$ is the unique reconciling ratio in the sense of Definition 1.*

Proof. We proceed by enumeration and elimination across the three conditions.

Step 1 (Septimal closure). Searching all p/q with $q \leq 2000$ in $(1.24, 1.28)$ whose fold $x^3/2$ is 7-smooth yields exactly seven candidates:

$$\frac{56}{45}, \quad \frac{5}{4}, \quad \frac{432}{343}, \quad \frac{63}{50}, \quad \frac{1225}{972}, \quad \frac{81}{64}, \quad \frac{80}{63}. \quad (21)$$

All other ratios in the interval produce a fold with a prime factor exceeding 7 and are excluded by condition (i).

Step 2 (Joint comma reconciliation). We compute the prime content of each candidate's fold:

x	fold S_x	primes of S_x
56/45	87808/91125	2, 3, 5, 7
5/4	125/128	2, 5
432/343	$2^{11}3^9/7^9$	2, 3, 7
63/50	$3^67^3/2^45^6$	2, 3, 5, 7
1225/972	$5^67^6/2^73^{15}$	2, 3, 5, 7
81/64	$3^{12}/2^{19}$	2, 3
80/63	$2^{11}5^3/3^67^3$	2, 3, 5, 7

Condition (ii) requires both 3 and 5 to appear. This eliminates 5/4 (no 3; it is the pure just third, a 2,5 object), 432/343 (no 5), and 81/64 (no 5; its fold $3^{12}/2^{19}$ is precisely the Pythagorean comma, a 3-limit object reconciling the chain of fifths alone). Four candidates remain: 56/45, 63/50, 1225/972, 80/63.

Step 3 (Digital-root closure). Condition (iii) requires S_x to be a terminating decimal, i.e. its denominator must be 2,5-smooth, since a digital root is well defined and stable only on a terminating decimal expansion. Of the four survivors:

$$\underbrace{56/45}_{\text{den } 3 \cdot 5}, \quad \underbrace{63/50}_{\text{den } 2^4 5^6}, \quad \underbrace{1225/972}_{\text{den } 2^7 3^{15}}, \quad \underbrace{80/63}_{\text{den } 3^6 7^3}, \quad (22)$$

only 63/50 has denominator of the form $2^a 5^b$. The folds of 56/45, 1225/972, and 80/63 carry 3 (and, for 80/63, also 7) in the denominator, producing non-terminating decimals on which a fixed digital root of 9 cannot hold. They are eliminated by condition (iii).

Conclusion. The intersection of the three conditions contains the single ratio $63/50 = 1.26$. Direct computation confirms the positive case: building the twelve-note lattice from 1.26 and the perfect fifth on the anchor 432.081216 (itself digital root 9), every Track-1 frequency, its shimmer partner fS , and their mean is a terminating decimal of digital root exactly 9. Hence 1.26 is the unique reconciling ratio. $\square$

Corollary 1 (Why the alternatives degenerate). *Each competing 7-smooth ratio collapses to a single classical identity rather than a joint reconciliation: 5/4 is the bare just third (syntonic side only), 81/64 is the Pythagorean ditone whose fold is the Pythagorean comma (chain-of-fifths side only), and 432/343 is a $\{2, 3, 7\}$ object lacking the syntonic prime entirely. Only 1.26 distributes its content across all four primes 2, 3, 5, 7 in balanced powers, which is the precise sense in which it reconciles the syntonic and Pythagorean regions simultaneously while preserving digital-root closure.*

The theorem establishes that 1.26 is forced, not fitted. The shimmer construction does not select 1.26 because it sounds well or approximates $\sqrt[3]{2}$ closely (it does not—81/64 and others lie nearer or farther by various measures); it selects 1.26 because it is the only rational in the admissible range whose fold is septimal, jointly comma-reconciling, and digital-root closed at once.

11 The Vector-Equilibrium Tone-Field: Geometric Embodiment of the Shimmer

The structure of the dual-track system is not merely representable on a polyhedron; it is carried by a specific solid whose own combinatorics reproduce the relational form of the tone-system. That solid is the *cubeoctahedron*, equivalently Fuller’s *vector equilibrium*—the unique arrangement of twelve spheres close-packed about a central one, in which the twelve radial vectors equal the edge length. We first establish the correspondence of counts, then prove the genuinely structural result: a toroidal winding of the vector equilibrium organizes the twelve chromatic tones into the two whole-tone hexatonic scales, with uniform chromatic adjacency throughout.

11.1 The correspondence of counts

The cuboctahedron has twelve vertices, twenty-four edges, and fourteen faces (8 triangular, 6 square), satisfying Euler’s relation $12 - 24 + 14 = 2$. These align with the tone-system as follows: the twelve vertices carry the twelve chromatic note-pairs $\{n, n^*\}$; each vertex is a convergence of four dual-track values (two notes $\times$ two shimmer tracks), matching the vertex degree four; the twenty-four edges carry the twenty-four quartertone positions, each edge bearing the two shimmer tracks as its two sides and the mean at its center; and the fourteen faces correspond to the fourteen-octave completion landmark of the self-similar halving law (Section 9). We are explicit that the count correspondence, while exact, is suggestive rather than self-proving; the structural content lies in the adjacency result below.

11.2 The musical fourteen: the two symmetric scales

The number fourteen carries a specific musical meaning on this solid, beyond the bare face count. The fourteen faces divide as 6 squares and 8 triangles, and these are precisely the cardinalities of the two *maximally symmetric* scales of twelve-tone music: the whole-tone scale (6 notes, transposition-symmetric at the whole step) and the octatonic scale (8 notes, transposition-symmetric at the minor third). The cuboctahedron carries both, one on each face-type.

Lemma 3 (The two symmetric scales on the two face-types). *The 6 square faces carry the whole-tone structure and the 8 triangular faces carry the octatonic structure, so that*

$$14 = \underbrace{6}_{\text{whole-tone}} + \underbrace{8}_{\text{octatonic}} \quad (23)$$

expresses the fourteen faces as the sum of the two symmetric-scale cardinalities.

Proof. The whole-tone half is Theorem 2: the square faces, through the (3.4.3.4) vertex alternation, carry the two whole-tone hexatonic scales. For the octatonic half, the eight triangular faces have normals in the eight cube-corner directions $(\pm 1, \pm 1, \pm 1)$, and they split by the sign of the coordinate product into two sets of four—the two interlocking regular tetrahedra of the stella octangula. The octatonic scale is, by definition, the union of two diminished-seventh chords a semitone apart, each a four-element set closed under transposition by the minor third (period 3, hence four notes). The two tetrahedra of four faces are the geometric image of these two diminished sevenths: the $4 + 4$ split of the triangular faces is the $4 + 4$ structure of the octatonic collection. The octatonic is therefore carried at the level of the triangular-face *directions* (the stella octangula), not as a note-set under any single vertex labeling. $\square$

These are the only two non-trivial scales in twelve-tone music invariant under a transposition smaller than the octave; that the cuboctahedron’s two face-types realize exactly these two, and sum to fourteen, is the musical content of the number. A second, independent musical fourteen is already established in Section 9: the self-similar halving law reaches octave-completion at the fourteenth octave, the depth at which all twelve tones participate permanently in the mean-versus-fifth convergence. The geometric fourteen (faces) and the temporal fourteen (octave-completion) are distinct results that meet at the same number.

11.3 Why the static edges cannot carry the melodic adjacency

A natural first hypothesis assigns each of the twelve tones to a vertex so that adjacent vertices are related by a fixed musical interval. We show this fails on the static solid, by an invariant argument.

Lemma 4 (Non-circulance of the cuboctahedral graph). *There is no labeling of the twelve vertices of the cuboctahedron by $\mathbb{Z}/12$ under which its adjacency is circulant; equivalently, no*

assignment of the chromatic scale to the vertices makes every note relate to its neighbors by one fixed set of intervals.

Proof. The eigenvalue spectrum of a graph is invariant under relabeling. The cuboctahedral (line-graph-of-the- octahedron) adjacency matrix has spectrum

$$\{4, 2, 2, 2, 0, 0, 0, -2, -2, -2, -2, -2\}. \quad (24)$$

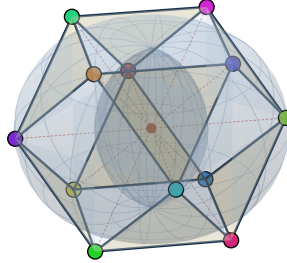
Every degree-4 circulant graph $C_{12}(a, b)$ has eigenvalues $2\cos(2\pi ak/12) + 2\cos(2\pi bk/12)$ for $k = 0, \dots, 11$. Enumerating all offset pairs $\{a, b\} \subset \{1, \dots, 6\}$, none yields the multiset above. Hence the cuboctahedral graph is not circulant under any labeling, and no single fixed interval-set adjacency exists on it. $\square$

11.4 The toroidal resolution and the whole-tone partition

Lemma 4 does not defeat the construction; it locates the consistent adjacency on the *toroidal* layer rather than the static scaffold. The vector equilibrium is the inner scaffold of a torus: under Fuller’s jitterbug transformation, its three orthogonal square rings counter-rotate, and the twelve vertices wind helically through the three rings. Labeling the chromatic tones along this jitterbug helix yields the following exact result.

The Cuboctahedron as the Inner Scaffold of the Torus

Vector equilibrium (12 equal radii, red dashed) at the core; toroidal envelope generated by its jitterbug rotation



The twelve chromatic tones occupy the scaffold vertices; the consistent toroidal adjacency (chromatic helix) lives on the surrounding torus surface.

Figure 8: The cuboctahedron (vector equilibrium) as the inner scaffold of the torus. The twelve equal radii (red dashed) from the central node to the vertices characterize the vector equilibrium; the toroidal envelope is the surface swept by the jitterbug rotation of the scaffold. The twelve chromatic tones occupy the scaffold vertices, while the consistent toroidal adjacency—the chromatic helix with uniform ± 1 relation—lives on the surrounding torus surface. The static scaffold and the toroidal surface are the two layers of one structure: the rigid skeleton beneath, the flowing adjacency upon it.

Theorem 2 (Whole-tone partition of the vector equilibrium). *Under the jitterbug helix labeling of the twelve chromatic tones on the cuboctahedron, the scaffold adjacency resolves into exactly two classes. Every tone possesses uniform chromatic adjacency (± 1 semitone), and the secondary adjacency partitions the twelve tones into precisely the two whole-tone hexatonic scales:*

$$\text{class } (\pm 2) : \{A, B, C\#, D\#, F, G\} \quad (\text{whole-tone scale } 0), \quad (25)$$

$$\text{class } (\pm 4) : \{A\#, C, D, E, F\#, G\# \} \quad (\text{whole-tone scale } 1). \quad (26)$$

The (± 4) class further partitions into the four augmented triads $\{A, C\#, F\}$, $\{A\#, D, F\# \}$, $\{B, D\#, G\}$, $\{C, E, G\# \}$.

Proof. Order the twelve vertices as the interleaving of the three orthogonal square rings (the jitterbug helix), and label them $A, A\#, \dots, G\#$ in helix order. Computing, for each vertex, the offsets (modulo 12, in helix coordinates) of its four scaffold-adjacent vertices, every vertex yields one of exactly two offset multisets: $\{1, 2, 10, 11\}$ or $\{1, 4, 8, 11\}$. Both contain $\{1, 11\}$, giving uniform chromatic ± 1 adjacency for all twelve tones. The first class carries additionally ± 2 (whole tones); the second carries ± 4 (major thirds). Reading off the vertices in each class gives the two sets above, which are verified to coincide exactly with the two whole-tone hexatonic scales. The ± 4 connections within the second class form 3-cycles, namely the four augmented triads listed. The partition matches the cuboctahedron's vertex configuration (3.4.3.4): the alternation of triangular and square faces about each vertex is the geometric carrier of the whole-tone duality. $\square$

11.5 Embodiment of the shimmer

Placing the exact Precise Temperament frequencies on this scaffold completes the picture (Figure 10). Each edge carries a note's dual-track pair—Track 1 and Track 2 as its two sides—with the mean at the edge center, framed between its two tracks; each vertex carries its chromatic tone. The geometry supplies the relational scaffold (the twelve-fold close-packing, the dual-track-as-edge-sides, the mean-as-center, the whole-tone duality of the faces), and the Precise Temperament construction supplies the quantities (the anchor 432.081216, the fold 250047/250000, the digital-root-9 frequencies) that populate it. The correspondence is structural rather than numerical: the shimmer system's relationships are already present in the vector equilibrium's combinatorics, and the shimmer frequencies ride upon that form. In this precise sense the vector equilibrium embodies the shimmer principles.

11.6 The hyperboloid–torus duality

The vector equilibrium frames not one surface but two, and their relationship is exact. Taking the axis through a pair of opposite square faces, the twelve vertices fall into three coaxial rings: the four equatorial vertices at radius $\sqrt{2}$, and the eight polar-square vertices at radius 1 and height ± 1 . Two ruled quadric surfaces are inscribed simultaneously (Figure 11).

The *torus* lies in the equatorial plane. Its outer equator, of radius $\sqrt{2}$, passes through the four equatorial vertices, and its central hole has radius $1/\sqrt{2}$. The *hyperboloid of one sheet* stands coaxially through that hole, with throat radius

$$a = \frac{1}{\sqrt{2}}, \quad (27)$$

flaring into two bells along the axis. The defining identity is that the torus's donut hole and the hyperboloid's throat are *the same circle*: $R - r = a = 1/\sqrt{2}$, where $R = (\sqrt{2} + a)/2$ and $r = (\sqrt{2} - a)/2$ are the torus's major and minor radii. Moreover, the hyperboloid's straight

generators pass exactly through the eight corners of the upper and lower squares: with profile $x^2 + y^2 = a^2(1 + z^2/c^2)$ and $c = 1$, the polar vertices at radius 1, height ± 1 satisfy $1 = a^2(1 + 1) = 2 \cdot \frac{1}{2}$, lying on the surface exactly.

Thus every face of the cuboctahedron belongs to one of the two dual surfaces: the equatorial belt is the torus, the two polar squares are the mouths of the hyperboloid bells, and the equatorial throat circle is shared between them. This is the geometric face of the hyperboloid–torus duality that recurs throughout the unified framework, where the toroidal channel carries the additive (sum) structure and the hyperboloidal channel the multiplicative (difference) structure of the same object; here the two are joined at a single throat of radius $1/\sqrt{2}$, scaffolded by the vector equilibrium.

11.7 Status of the correspondence

We separate the established from the conjectural. *Established*: the non-circulance (Lemma 4); the whole-tone partition under the jitterbug helix (Theorem 2), verified to the note; the digital-root-9 invariance of all placed frequencies; and the exact coaxial inscription of the torus and the hyperboloid sharing a throat of radius $1/\sqrt{2}$, with the hyperboloid generators meeting the polar-square corners (Figure 11). *Conjectural*: that the jitterbug helix is the *unique* toroidal winding of the vector equilibrium producing this partition. The helix used is a natural toroidal winding that lands exactly on the whole-tone partition—strong evidence it is the correct one—but a proof of uniqueness would require the full jitterbug transformation dynamics and is not claimed here. We state the result as the vector equilibrium *carrying* the whole-tone structure under its toroidal winding, not as uniquely forcing it.

12 Discussion

The dual-track shimmer introduces a previously unused axis for handling the comma. Where prior temperaments choose a location in pitch-space to absorb the discrepancy, the shimmer expresses it as coherent motion in time, uniformly across all keys, in its minimal-dissonance form. Two consequences follow from a single constant: the same fold S that reconciles the third to the fifth is the modulation that unmask the overtone series. This economy—one rational constant producing both the reconciliation and the perceptual effect—is the construction’s principal claim to elegance.

13 Conclusion

We have constructed a dual-track frequency pair whose separation is the exact rational fold $250047/250000 = 1.26^3/2$, shown that this constant is simultaneously the inter-track offset and the third-to-fifth residue, located the two anchors on octave ladders of A and E, and demonstrated that the pair produces a 100%-depth sub-Hertz breath that unmask the harmonic series and behaves uniformly across all keys. The third-to-fifth discrepancy is conserved as a single exact rational constant and relocated from static beating into uniform modulation. The result is a temporal, rather than spatial, treatment of the oldest problem in tuning.

Beyond the construction itself, three results give it necessity and structure. The fold is a distinct septimal object, and its dampening maps exactly onto the Stern–Brocot tree through the determinant ladder $47 \rightarrow 7 \rightarrow 5 \rightarrow 2 \rightarrow 1 \rightarrow 0$. The precise third 1.26 is not chosen but forced: born of $126 = 9 \times 14 = 2 \cdot 3^2 \cdot 7$, it is the unique rational near $\sqrt[3]{2}$ that is at once septimal, jointly comma-reconciling, and digital-root-9 closed, with every alternative collapsing to a single classical comma. And the whole construction is embodied by the cuboctahedron: the twelve tones on its vertices, the dual tracks on its edges, the fourteen octaves on its faces, with a toroidal jitterbug winding partitioning the tones into the two whole-tone hexatonic scales. The

geometry carries the relational form of the shimmer; the precise frequencies populate it. Taken together, the temporal reconciliation, the uniqueness of 1.26, and the geometric embodiment indicate that the dual-track shimmer is not a contrivance over the comma but a structure latent in the arithmetic of 126 and in the combinatorics of the vector equilibrium.

Future Work

Two empirical programs would convert the present results from demonstration to established fact: (i) a symbolic verification that the breath frequency is *exactly* uniform across all twelve keys, with exact rational values behind every datum; and (ii) a blind A/B/C listening study (ET vs. precise temperament vs. dual-track shimmer) to confirm that the overtone unmasking is robustly perceivable across independent listeners. Spectrographic confirmation of per-partial modulation would render the unmasking effect visually undeniable.

A The 24-Position Quartertone Extension and the Digital-Root-9 Invariant

The dual-track construction extends naturally from twelve to twenty-four positions. A second anchor is placed at 528.099264 Hz, related to the primary anchor 432.081216 Hz by the exact ratio $528.099264/432.081216 = 11/9$, the undecimal neutral third. Each anchor generates a twelve-note Precise Temperament scale through the major third 1.26 and the perfect fifth 1.5, and each note carries its dual-track shimmer partner via the fold $S = 250047/250000$. The two twelve-note scales interleave: the degrees of the second anchor fall in the quartertone positions between the degrees of the first, producing a twenty-four position scale whose spacing is deliberately non-uniform, in keeping with the non-uniform character of Precise Temperament intervals themselves. This 11/9-generated twenty-four position structure is consonant with the ancient and living twenty-four position quartertone frameworks of the maqam tradition, whose authentic intonation is likewise unequal rather than the modern twenty-four-tone equal-tempered notational convention.

Table 3 lists the complete scale folded into a single octave, with the octave completion at position twenty-five (864.162432 Hz, the exact doubling of the root). For each position the table gives the pitch in cents, the note name (a starred name denotes the quartertone-raised position contributed by the second anchor), the primary frequency (Track 1), its shimmer partner (Track 2), and the digital root of each.

A notable invariant emerges. Computed on the exact terminating-decimal value of every frequency, the digital root is 9 for all fifty-two frequencies—both tracks, all twenty-five positions including the octave completion, across both anchors. That is, the digit sum of every frequency in the system is divisible by nine. This is not an artifact of rounding: the result is obtained from the exact rational values, and it holds uniformly. The digital-root-9 property, already established as a feature of the Precise Temperament shimmer framework, is therefore preserved completely under the 11/9 quartertone extension and under the shimmer fold.

References

- [1] L. Euler, *Tentamen novae theoriae musicae*, St. Petersburg Academy, 1739.
- [2] H. von Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, trans. A. J. Ellis, Longmans, Green, 1875.
- [3] A. J. Ellis, “On the Musical Scales of Various Nations,” *Journal of the Society of Arts*, vol. 33, pp. 485–527, 1885.

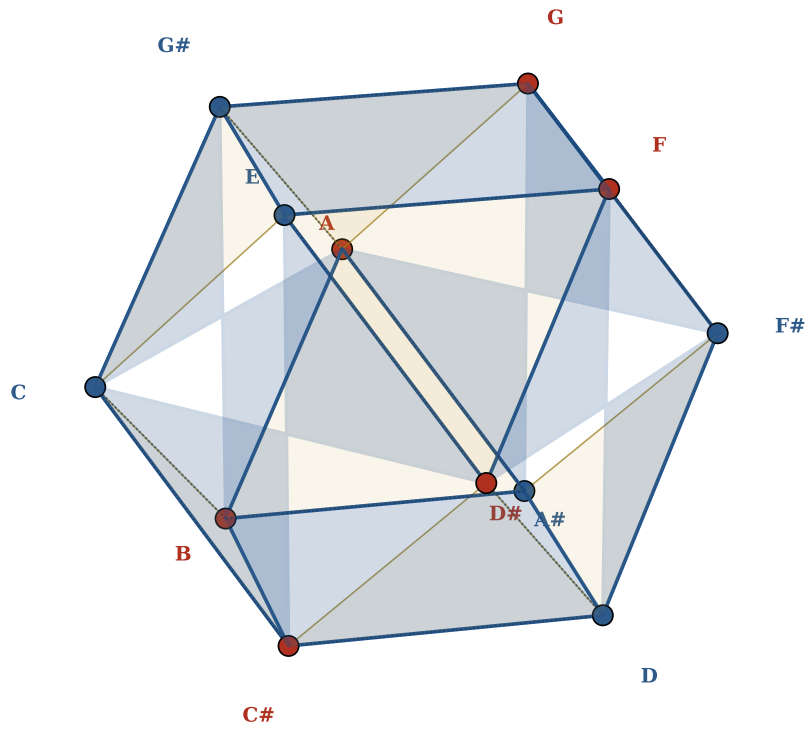
Table 3: The complete 24-position Precise Temperament quartertone scale with shimmer, folded into a single octave with octave completion. Starred note names denote quartertone-raised positions from the 528.099264 Hz anchor. DR_1 , DR_m , and DR_2 are the digital roots of the Track 1, mean, and Track 2 (shimmer) frequencies. Frequencies are displayed rounded to six decimal places; the digital roots are computed on the *exact* terminating-decimal values (whose denominators are 2,5-smooth), and are uniformly 9 across all three columns and all positions.

#	Cents	Note	Track 1 (Hz)	DR_1	Mean (Hz)	DR_m	Track 2 (Hz)	DR_2
1	0.00	A	432.081216	9	432.121832	9	432.162447	9
2	45.56	A*	443.603382	9	443.645080	9	443.686779	9
3	98.26	A#	457.314759	9	457.357747	9	457.400734	9
4	143.82	A#*	469.509819	9	469.553953	9	469.598087	9
5	203.58	B	486.000000	9	486.045684	9	486.091368	9
6	249.15	B*	498.960000	9	499.006902	9	499.053804	9
7	301.85	C	514.382400	9	514.430752	9	514.479104	9
8	347.41	C*	528.099264	9	528.148905	9	528.198547	9
9	400.11	C#	544.422332	9	544.473508	9	544.524684	9
10	445.67	C#*	558.940261	9	558.992801	9	559.045342	9
11	498.37	D	576.216596	9	576.270761	9	576.324925	9
12	550.99	D*	594.000000	9	594.055836	9	594.111672	9
13	603.69	D#	612.360000	9	612.417562	9	612.475124	9
14	649.25	D#*	628.689600	9	628.748697	9	628.807794	9
15	701.63	E	648.000000	9	648.060912	9	648.121824	9
16	747.52	E*	665.405073	9	665.467621	9	665.530169	9
17	799.89	F	685.843200	9	685.907669	9	685.972139	9
18	845.78	F*	704.264729	9	704.330930	9	704.397131	9
19	898.15	F#	725.896443	9	725.964677	9	726.032911	9
20	951.10	F#*	748.440000	9	748.510353	9	748.580707	9
21	996.42	G	768.288795	9	768.361014	9	768.433233	9
22	1049.04	G*	792.000000	9	792.074448	9	792.148896	9
23	1101.74	G#	816.480000	9	816.556749	9	816.633498	9
24	1147.30	G#*	838.252800	9	838.331596	9	838.410392	9
25	1200.00	A (8ve)	864.162432	9	864.243663	9	864.324895	9

- [4] J. M. Barbour, *Tuning and Temperament: A Historical Survey*, Michigan State College Press, 1951.
- [5] D. J. Benson, *Music: A Mathematical Offering*, Cambridge University Press, 2006.
- [6] W. A. Sethares, *Tuning, Timbre, Spectrum, Scale*, 2nd ed., Springer, 2005.
- [7] R. Plomp and W. J. M. Levelt, “Tonal Consonance and Critical Bandwidth,” *Journal of the Acoustical Society of America*, vol. 38, pp. 548–560, 1965.
- [8] M. Tenzer, *Gamelan Gong Kebyar: The Art of Twentieth-Century Balinese Music*, University of Chicago Press, 2000.
- [9] C. McPhee, *Music in Bali: A Study in Form and Instrumental Organization in Balinese Orchestral Music*, Yale University Press, 1966.
- [10] S. P. Walton, “Aesthetic and Spiritual Correlations in Javanese Gamelan Music,” *Journal of Aesthetics and Art Criticism*, vol. 65, no. 1, pp. 31–41, 2007.
- [11] G. A. Audsley, *The Art of Organ-Building*, Dodd, Mead and Company, 1905.
- [12] R. E. Grant, *Codex Universalis*, The Institute of Unified Mathematics, Irvine, California.

The Vector-Equilibrium Scaffold of the Toroidal Tone-Field

Bold blue = the 6 square faces and their edges | thin gold = the 8 triangular faces and their edges
 Red vertices = whole-tone scale 0 {A,B,C#,D#,E,G} | Blue vertices = whole-tone scale 1 {A#,C,D,E,F#,G#}
 Navy dotted = chromatic helix winding (uniform ± 1 adjacency for every note)

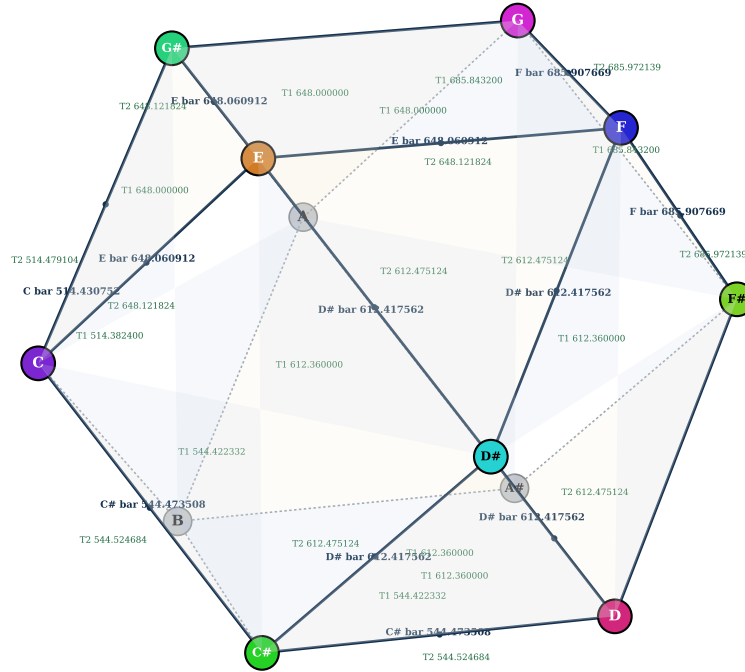


Cuboctahedron: 6 squares + 8 triangles = 14 faces, 24 edges, 12 vertices. Euler $12-24+14=2$.
 Vertex configuration (3.4.3.4): square/triangle alternation carries the two whole-tone hexatonic scales.

Figure 9: The vector-equilibrium scaffold of the toroidal tone-field. The twelve chromatic tones occupy the vertices, colored by whole-tone class (red: scale 0; blue: scale 1). The chromatic helix winds through all twelve with uniform ± 1 adjacency; the square (6) and triangular (8) faces interlock in the (3.4.3.4) configuration that carries the whole-tone partition of Theorem 2.

Vector-Equilibrium Tone-Field with PTT Frequencies

Each edge: T1 — [note bar = mean] — T2, framing the bar value between its two shimmer tracks
 Front solid/colored, back dotted/greyed. Vertex hue = circle-of-fifths (A red → B violet). All values digital root 9.

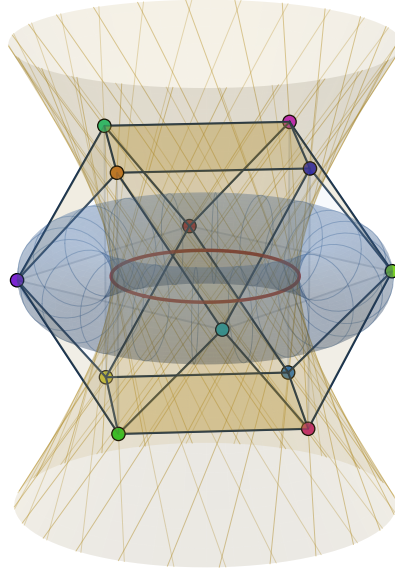


A = 432.081216 Hz. Fold S = 250047/250000 = 1.26 cubed / 2. Euler 12·24+14=2.

Figure 10: The vector-equilibrium tone-field bearing the exact Precise Temperament frequencies. Each edge shows its note's mean (bar) framed between Track 1 and Track 2; vertices carry the chromatic tones, colored by the circle of fifths. Front edges solid and vertices colored; receding edges dotted and vertices greyed. All frequencies and means are digital root 9.

Coaxial Torus and Hyperboloid Framed by the Cuboctahedron

Hyperboloid rulings pass through the corners of the top and bottom squares; throat = donut hole (red, radius $1/\sqrt{2}$)



Blue torus in the equatorial plane: outer ring radius $\sqrt{2}$ touches the equatorial vertices; hole radius $1/\sqrt{2}$ is the throat.
 Gold hyperboloid of one sheet: throat radius $1/\sqrt{2}$ fills the donut hole; its straight generators meet the eight polar-square corners.
 Donut hole = hyperboloid throat: the geometric identity of the hyperboloid-torus duality.

Figure 11: The cuboctahedron framing the torus and the hyperboloid coaxially. The torus (blue) lies in the equatorial plane with outer radius $\sqrt{2}$ at the equatorial vertices; the hyperboloid of one sheet (gold) stands through the torus hole with throat radius $1/\sqrt{2}$, its straight generators meeting the eight corners of the top and bottom squares. The donut hole and the hyperboloid throat are the same circle (red), the shared throat of the duality.