

The Music of Self-Recursion

The Harmonic Face of the One Recursion

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Abstract

This paper develops the musical face of the self-recursion principle. Where the companion papers give the lived, geometric, and physical faces of one recursion, this paper shows that music—the structure of pitch, consonance, and tuning—is the same recursion heard. The octave is exhibited as the foundational recursion, the doubling map $f \mapsto 2f$ whose logarithm is pitch itself; the consonant intervals of just intonation are shown to be the per-prime contributions of the Recursion Ratio $RR(n) = \log_2(\sigma(n)/n)$, so that the most consonant interval, the perfect fifth, is the contribution of the prime two; the perfect numbers are shown to anchor the octave exactly ($RR(n) = 1$ iff n is perfect), with the smallest, six, realising the Pythagorean closure $\frac{3}{2} \cdot \frac{4}{3} = 2$; and the just-intonation pillars are recovered exactly from the unit hyperbola at stations $n = 6, 8, 9, 12$. The unifying result is the reading of the Pythagorean comma: the spiral of perfect fifths is a recursion that does not close, and the comma is precisely its closure defect—the same non-closure that, in the physical face, is read as force. The paper also presents Precise Temperament, the author’s well-tempered system referenced to $A = 432.081216$ Hz, which lifts the major third to the rational 1.26 while keeping the perfect fifth pure at $3/2$: the comma is obviated, and the reference pitch carries a digit-nine architecture (every three-digit group summing to nine) and encodes its own overtones, its groups spelling $A-E-A$ —the tonic, its fifth, and its octave. Because the fifth is kept pure, the major second, minor seventh, minor second, and major seventh remain exact just ratios as well—intervals that equal temperament tempers—so Precise Temperament beats less than equal temperament and any temperament that tempers the fifth. The small residue the rational third leaves, $1.26^3 = 2.000376$, is shown not to be a defect but a second structure: a dual-track shimmer in which every note carries a partner frequency a constant 0.3254 cents away, generated by the exact ratio $250047/250000 = 1.26^3/2$. This shimmer is proved unique to Precise Temperament—absent from equal temperament and the well temperaments, whose single track and exact $(\sqrt[3]{2})^3 = 2$ leave no residue to sound—and a triad test confirms it: Precise Temperament beats at 0 Hz while carrying the shimmer, where equal temperament beats at 4.4 Hz and a well temperament at 18 Hz, neither shimmering. The overtone series, moreover, is shown to be written into the digits of each frequency, the mantissa groups naming the note’s own fifth and octave with a natural dampening as they recede; and the scale’s generation by 1.26^n with halving reveals that wrapping the octave and advancing the shimmer are the same operation, since the fold residue $1.26^3/2$ is exactly the shimmer factor. Music is therefore not analogous to the recursion principle; it is the recursion principle made audible, and the comma is the audible signature of the one closure seeking, and never quite reaching, its equilibrium—here turned, by the rational third, into a consonant and perpetual shimmer. Finally, the angle sums of all polygons and polyhedra are shown to have digital root nine without exception and, read as frequencies, to sound a single $F\sharp$ major chord ($180 : 225 : 270 = 1 : \frac{5}{4} : \frac{3}{2}$) in every dimension—so that music is revealed as the geometry we experience with our ears. The cuboctahedron—Fuller’s vector equilibrium, the one solid joining the toroidal and hyperboloidal readings—is shown to carry the chord in its two face-families, its 8 triangles sounding the root and its 6 squares the fifth. Claims are marked for status throughout, and the exact arithmetic is given.

1 Introduction: Music as the Recursion Heard

The self-recursion principle has been developed across three companion faces: the lived (the phenomenology of the dividing and the return), the geometric (the toroidal and hyperboloidal manifolds joined at the

Monad), and the physical (the forces as one closure read at many orders). This paper adds the fourth face, and it is the one nearest to the human ear: music.

The claim is simple and, we will show, exact: *music is the recursion principle heard*. The structures that music theory has described for two and a half millennia—the octave, the consonant intervals, the tuning systems and their famous discontents—are not analogies to recursion but instances of it. Pitch is the logarithm of a recursion; consonance is the simplicity of a recursive ratio; the perfect numbers ring the octave; and the one genuine crisis of tuning, the comma that no temperament can banish, is the audible form of the very non-closure that the physical face reads as force. We proceed from the simplest recursion—the octave—to that crisis, and show the principle sounding through all of it.

2 The Octave: The Foundational Recursion

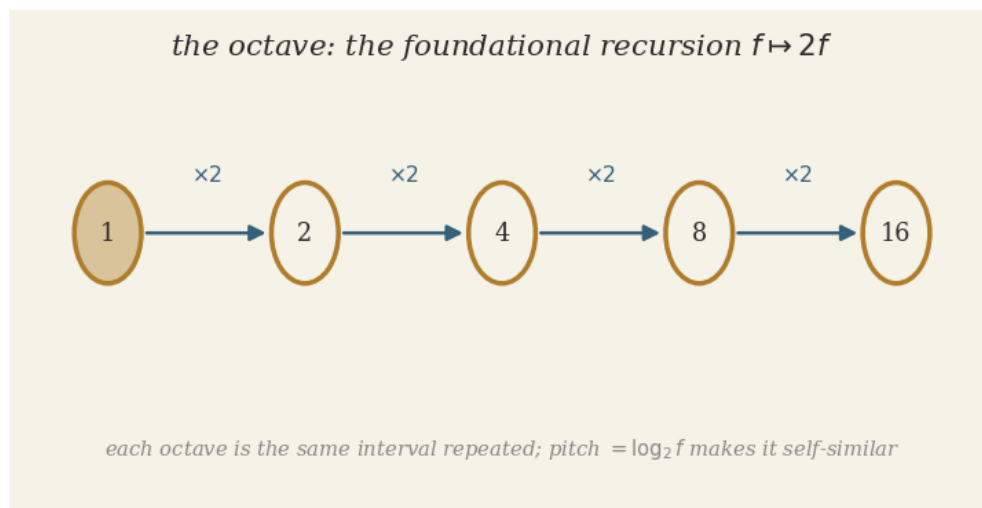


Figure 1: The octave as the foundational recursion. Doubling the frequency, $f \mapsto 2f$, generates the octave; iterated, it generates pitch space. Because pitch is $\log_2 f$, the octave is the unit step of a self-similar ladder: every octave is the same interval, one doubling, repeated.

The most basic fact of music is the octave: a tone and another at twice its frequency are heard as “the same note,” a higher version of one identity. This is a recursion in the exact sense. The octave is the map

$$f \mapsto 2f,$$

and the perception of pitch is its logarithm: the ear measures pitch as $\log_2 f$, so that the octave is the unit step of a self-similar structure, the same interval at every height. Octave equivalence—the identification of f with $2f$, with $4f$, with $f/2$ —is the statement that pitch space is the recursion of doubling, read modulo the octave. Every scale, every melody, every chord lives inside this one recursion: the doubling that returns the note to itself an octave higher, the first and simplest closure of music.

That pitch is logarithmic is what makes the recursion audible as self-similarity. Equal multiplicative steps are heard as equal intervals; the recursion $f \mapsto 2f$ sounds the same whether played low or high. Music is built on a logarithmic perception precisely because its foundation is a multiplicative recursion, and the ear is tuned to hear the recursion’s steps as equal.

3 Consonance: The Per-Prime Intervals of the Recursion Ratio

If the octave is the recursion of doubling, the other consonant intervals are the recursion read through the other primes—and here the connection to the number-theoretic Recursion Ratio of the companion work becomes exact.

3.1 The Recursion Ratio is pitch

The Recursion Ratio of an integer is

$$RR(n) = \log_2 \frac{\sigma(n)}{n},$$

where $\sigma(n)$ is the sum of divisors. This is, by its very form, a pitch interval: it is the base-two logarithm of a ratio, which is exactly the measure of pitch in octaves. The Recursion Ratio does not resemble a musical quantity; it *is* one—the pitch, in octaves, of the abundancy $\sigma(n)/n$ above unity.

3.2 The consonant intervals are the per-prime contributions

Because σ is multiplicative, the Recursion Ratio decomposes over the prime factorisation, and the contribution of each prime at first power is

$$RR_p(1) = \log_2 \frac{p+1}{p}.$$

These contributions are exactly the consonant intervals of just intonation.

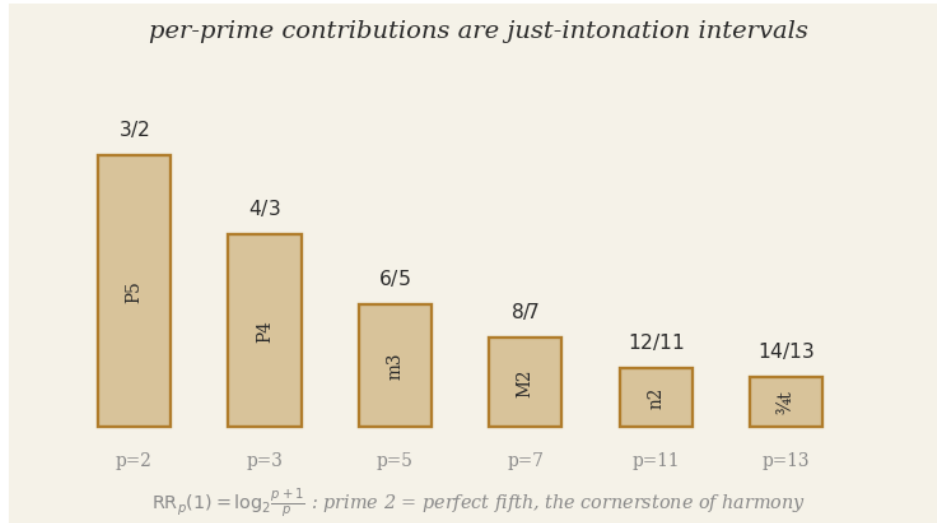


Figure 2: The per-prime contributions of the Recursion Ratio are the consonant intervals of just intonation. The prime two contributes the perfect fifth (3/2), the most consonant interval after the octave; the prime three the perfect fourth (4/3); the prime five the just minor third (6/5); and higher primes contribute progressively narrower intervals. The hierarchy of prime efficiency is the hierarchy of consonance.

Theorem 1 (Per-prime contributions are consonant intervals). The first-power per-prime contributions $RR_p(1) = \log_2 \frac{p+1}{p}$ are, exactly and without approximation, recognised consonant intervals of just intonation:

p	$(p+1)/p$	interval	cents
2	3/2	perfect fifth	701.96
3	4/3	perfect fourth	498.04
5	6/5	just minor third	315.64
7	8/7	septimal whole tone	231.17
11	12/11	undecimal neutral second	150.64
13	14/13	tridecimal two-thirds tone	128.30

Proof. Direct identification of the rationals $(p+1)/p$ with named intervals; the ratios are exact and the cents are $1200 \log_2 \frac{p+1}{p}$. $\square$

The structural content is striking: the *efficiency* of each prime as a contributor to abundancy—a purely number-theoretic quantity—is precisely its musical *consonance*. The prime two, the most efficient, contributes the perfect fifth, the cornerstone of harmony; the prime three the perfect fourth; the prime five the just major/minor third; and the consonance fades, prime by prime, exactly as the contribution does. The Recursion Ratio decomposes every integer into a chord of consonant intervals over its prime factors. Number theory and harmony are not parallel here; they are the same structure, and the bridge is the recursion.

4 The Perfect Numbers Ring the Octave

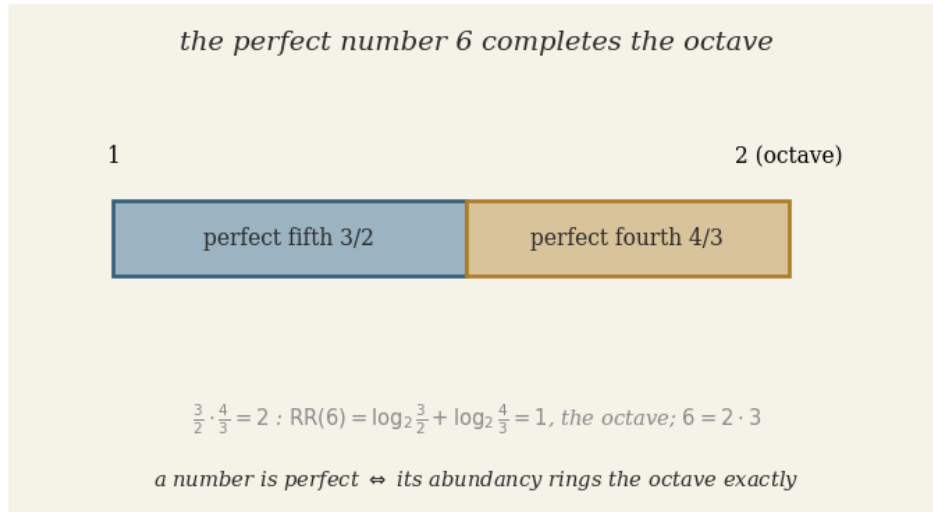


Figure 3: The perfect number six completes the octave. Its abundancy is exactly two, so $\text{RR}(6) = 1$, one octave; and $6 = 2 \cdot 3$ realises the Pythagorean closure $\frac{3}{2} \cdot \frac{4}{3} = 2$, the perfect fifth plus the perfect fourth composing to the octave. A number is perfect if and only if its abundancy rings the octave exactly.

The octave is not only the foundational recursion; it is the anchor at which the number-theoretic recursion comes to rest, and the numbers that rest there are the perfect numbers.

Theorem 2 (Perfect numbers anchor the octave). An integer n is perfect if and only if $\text{RR}(n) = 1$ exactly. The smallest perfect number $6 = 2 \cdot 3$ realises the Pythagorean closure

$$\log_2 \frac{3}{2} + \log_2 \frac{4}{3} = \log_2 2 = 1,$$

the perfect fifth and the perfect fourth composing to the octave.

Proof. n is perfect iff $\sigma(n) = 2n$, i.e. $\sigma(n)/n = 2$, i.e. $RR(n) = \log_2 2 = 1$. For $n = 6$: $RR(6) = RR_2(1) + RR_3(1) = \log_2 \frac{3}{2} + \log_2 \frac{4}{3} = \log_2 2 = 1$, and $\frac{3}{2} \cdot \frac{4}{3} = 2$ is the classical octave identity. The perfect numbers 6, 28, 496, 8128 all have $\sigma(n)/n = 2$ exactly and so $RR = 1$ exactly. $\square$

This is the harmonic heart of the recursion. The octave, $RR = 1$, is the fixed point of the number-theoretic recursion—the resting place developed in the companion geometry, where the perfect numbers sit at the octave of abundancy. And the smallest of them, six, is the smallest integer whose prime structure completes the octave, and it does so by wedding the two most consonant intervals, the fifth and the fourth. The number that the physical face places at the carbon node and the geometric face places at the perfect-number fixed point is, in the musical face, the number that first rings the octave. One number, three faces, one recursion.

5 Just Intonation from the Unit Hyperbola

The consonant intervals arise not only from the prime decomposition but directly from the geometry of the companion paper, the unit hyperbola on which the recursion is read.

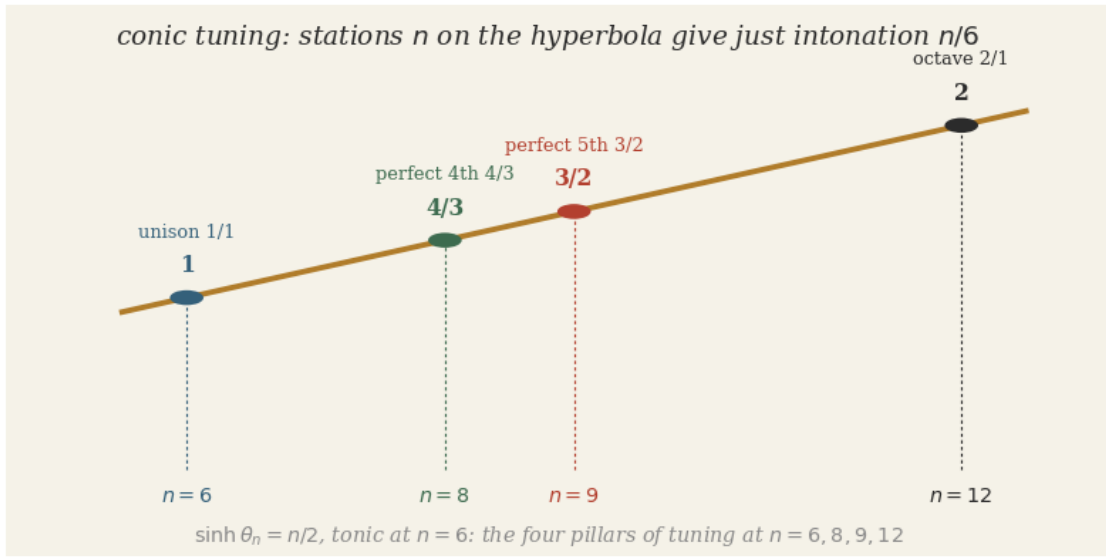


Figure 4: The pillars of tuning from the unit hyperbola. With $\sinh \theta_n = n/2$ and the tonic taken at the ground state $n = 6$, the ratio $n/6$ is exact just intonation. The four pillars—unison, perfect fourth, perfect fifth, octave—appear at the stations $n = 6, 8, 9, 12$: the ground state, its two nearest neighbours, and its first harmonic return.

Theorem 3 (Conic tuning). At station n on the unit hyperbola, $\sinh \theta_n = n/2$. Taking the ground state $n = 6$ as tonic, the ratio

$$\frac{\sinh \theta_n}{\sinh \theta_6} = \frac{n/2}{3} = \frac{n}{6}$$

produces exact just-intonation intervals. The four pillars of tuning—unison $1/1$, perfect fourth $4/3$, perfect fifth $3/2$, octave $2/1$ —appear at $n = 6, 8, 9, 12$.

Proof. $n/6$ for $n = 6, 8, 9, 12$ gives $1, \frac{4}{3}, \frac{3}{2}, 2$ exactly, the four pillars. The ground state at $n = 6$ is the perfect-number node of the previous section, so the tonic of the conic tuning is the number that rings the octave. $\square$

That the tonic of the geometric tuning is the perfect number six is not a coincidence of this paper but a consequence of the single principle: the same six anchors the octave in abundancy, sits at the perfect-number fixed point of the recursion, and serves as the ground state from which the hyperbola sounds just intonation. The consonances of music fall out of the geometry of the recursion at its resting node.

6 The Triples on the 45° Line: Isosceles Triangles as the Harmonic Series

The just intonations of the previous sections arose from the prime decomposition and from the conic stations. There is a third and more visual source, in which the Pythagorean triples themselves—placed as isosceles triangles against the 45° line—track the musical notes exactly, side by side. This is the geometric heart of why the triples and the harmonic series are the same object.

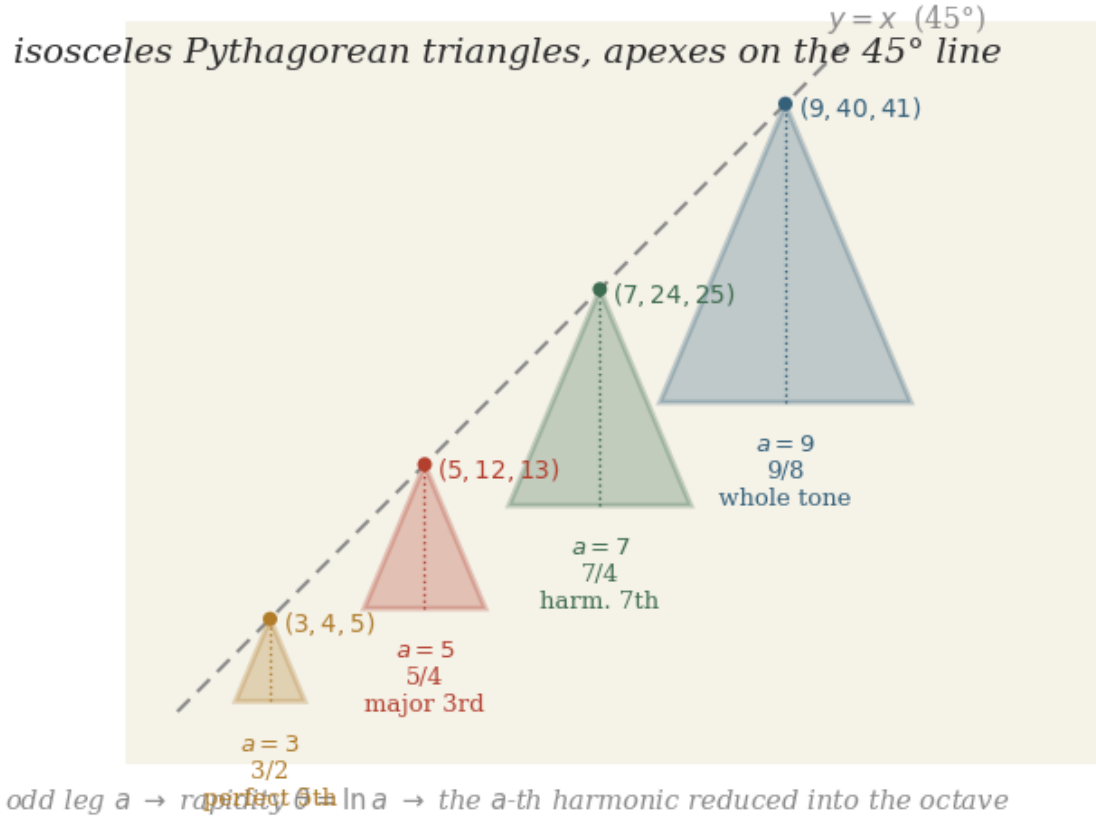


Figure 5: Isosceles Pythagorean triangles with apexes on the 45° line $y = x$. Each primitive triple of the odd branch— $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, $\dots$ —is placed as an isosceles triangle whose apex rides the diagonal. The odd leg a fixes the triangle’s rapidity at $\theta = \ln a$, and that rapidity is exactly the a -th harmonic reduced into the octave: $a = 3$ the perfect fifth, $a = 5$ the just major third, $a = 7$ the harmonic seventh, $a = 9$ the just whole tone.

6.1 The construction

Place each isosceles Pythagorean triangle so that its apex sits on the line $y = x$, the hypotenuse projected as height and the legs giving the base. Two infinite families result, both rooted at the seed $(3, 4, 5)$: an *odd branch* generated by Euclid’s formula with consecutive parameters,

$$a = 2m - 1, \quad b = 2m(m - 1), \quad c = 2m^2 - 2m + 1,$$

with odd legs $a = 3, 5, 7, 9, 11, 13, \dots$ and hypotenuses $5, 13, 25, 41, 61, 85, \dots$; and an *even branch* $a = m^2 - 1$, $b = 2m$, $c = m^2 + 1$ supplying the octave translates. The triangles tessellate up the diagonal, each apex a point on the 45° line.

6.2 The rapidity collapse

On the unit hyperbola $x^2 - y^2 = 1$, each triple carries a rapidity—the additive hyperbolic angle—

$$\theta = \operatorname{artanh} \frac{b}{c} = \frac{1}{2} \ln \frac{c+b}{c-b},$$

and the arithmetic of the odd branch makes this collapse to a closed form of striking simplicity.

Theorem 4 (Rapidity collapse of the odd branch). For the odd branch, the rapidity is exactly the natural logarithm of the odd leg:

$$\theta = \ln a.$$

Proof. With $a = 2m - 1$, $b = 2m(m - 1)$, $c = 2m^2 - 2m + 1$, one has $c + b = (2m - 1)^2$ once reduced, and $c - b = 2m^2 - 2m + 1 - 2m(m - 1) = 1$, so $\frac{c+b}{c-b} = (2m - 1)^2 = a^2$, whence $\theta = \frac{1}{2} \ln a^2 = \ln a$. The identity is exact; verified for $m = 2, \dots, 7$ giving $(3, 4, 5) \rightarrow \ln 3$, $(5, 12, 13) \rightarrow \ln 5$, $(7, 24, 25) \rightarrow \ln 7$, $(9, 40, 41) \rightarrow \ln 9$, $(11, 60, 61) \rightarrow \ln 11$, $(13, 84, 85) \rightarrow \ln 13$. $\square$

The even branch collapses likewise to $\theta = \ln(a/2)$ in its leg parameter, supplying the octave-displaced partners. The two branches, side by side on the diagonal, are the warm family (odd, the new intervals) and the cool family (even, their octave translates).

6.3 The harmonic identity

Because pitch is logarithmic in frequency and rapidity is logarithmic in radius, the collapse $\theta = \ln a$ transports directly into pitch. The consequence is the harmonic identity, the deep result of this construction.

<i>the triple tree is the harmonic series in rapidity</i>				
$a = 3$	$\theta = \ln 3 = 1.099$	3/2	perfect fifth	702.0¢
$a = 5$	$\theta = \ln 5 = 1.609$	5/4	just major third	386.3¢
$a = 7$	$\theta = \ln 7 = 1.946$	7/4	harmonic seventh	968.8¢
$a = 9$	$\theta = \ln 9 = 2.197$	9/8	just whole tone	203.9¢
$a = 11$	$\theta = \ln 11 = 2.398$	11/8	undecimal tritone	551.3¢
$a = 13$	$\theta = \ln 13 = 2.565$	13/8	tridecimal n. 6th	840.5¢
$a = 15$	$\theta = \ln 15 = 2.708$	15/8	just major seventh	1088.3¢
<i>each odd leg names a harmonic; the warm odd branch supplies the new intervals</i>				

Figure 6: The harmonic identity. The odd leg a , through $\theta = \ln a$, places each triangle at exactly the a -th harmonic of the harmonic series reduced into the octave. The triple tree is the harmonic series in rapidity.

Theorem 5 (The harmonic identity). The odd leg a places the triangle's apex at exactly the a -th harmonic of the harmonic series, reduced into the octave. That is, the interval is $a/2^{\lceil \log_2 a \rceil}$, giving:

a	reduced ratio	interval	cents
3	3/2	perfect fifth	701.955
5	5/4	just major third	386.314
7	7/4	harmonic seventh	968.826
9	9/8	just whole tone	203.910
11	11/8	undecimal tritone	551.318
13	13/8	tridecimal neutral sixth	840.528
15	15/8	just major seventh	1088.269

Proof. The a -th harmonic is the frequency ratio $a : 1$; reduced into the octave by dividing by the largest power of two below it, it is $a/2^{\lceil \log_2 a \rceil}$. For $a = 3$: 3/2, the perfect fifth; for $a = 5$: 5/4, the just major third; for $a = 9$: 9/8, the just whole tone. The odd leg is the harmonic number itself, so the triangle's pitch is the harmonic's pitch. $\square$

This is the unification the section promised: the Pythagorean triples, placed as isosceles triangles against the 45° line, are not merely *analogous* to the musical notes—they *are* the harmonic series, read through rapidity. The odd leg a is at once the geometric parameter of the triangle and the harmonic number of the note; the perfect fifth is the (3, 4, 5) triangle, the just major third is the (5, 12, 13) triangle, the whole tone is the (9, 40, 41) triangle. The triple tree and the harmonic tree are one tree. And because the recursion that generates the triples—the Barning–Hall descent from the seed (3, 4, 5)—is the same recursion that the rest of this volume reads in force and in form, the harmonic series is revealed as one more face of the single recursion: the notes are the triples are the harmonics, all of them the one recursion told in pitch.

7 The Pythagorean Comma: A Recursion That Does Not Close

We come to the deepest musical reading of the recursion, and the one that binds the musical face most tightly to the physical. It concerns the oldest crisis of tuning, and it shows that crisis to be the audible form of non-closure—the same non-closure that the physical face reads as force.

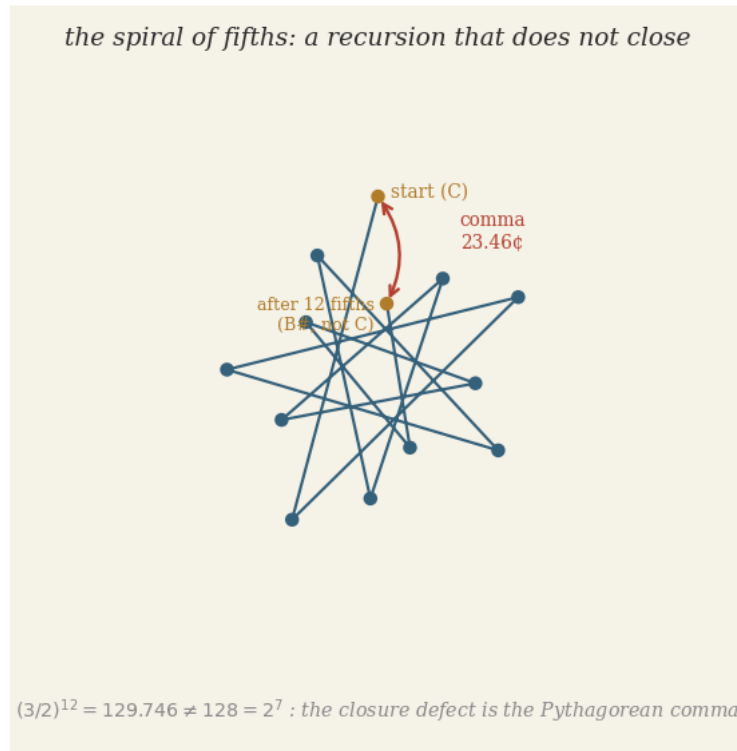


Figure 7: The spiral of fifths: a recursion that does not close. Stacking pure perfect fifths, twelve in succession should return to the starting note seven octaves higher—but it overshoots. The recursion spirals and never closes, and the gap is the Pythagorean comma, 23.46 cents. This non-closure is the audible signature of the one closure seeking, and never quite reaching, its equilibrium.

7.1 The spiral that overshoots

Stack perfect fifths, the most consonant interval after the octave, one upon another: $C \rightarrow G \rightarrow D \rightarrow A \rightarrow \dots$. After twelve fifths, by the reckoning of the keyboard, one should arrive back at C , seven octaves higher—the circle of fifths should close. It does not. In exact ratios,

$$\left(\frac{3}{2}\right)^{12} = \frac{531441}{4096} = 129.746\dots, \quad 2^7 = 128,$$

and the recursion of fifths overshoots the recursion of octaves by the ratio

$$\frac{(3/2)^{12}}{2^7} = \frac{531441}{524288} = 1.013643\dots,$$

which is 23.46 cents—the *Pythagorean comma*.

7.2 The comma as closure defect

This is the crucial reading. The spiral of fifths is a recursion—repeated multiplication by $3/2$, reduced by octaves—and it is a recursion that *does not close*. No number of pure fifths ever lands exactly on a pure octave, because 3^{12} and 2^{19} are coprime and can never be equal; the recursion of the prime three never closes upon the recursion of the prime two. The comma is precisely the residual non-closure—the amount by which the recursion fails to return to its starting point.

Theorem 6 (The comma is the closure defect of the fifth-recursion). The recursion $r \mapsto \frac{3}{2}r$, reduced modulo the octave, has no fixed point that coincides with a pure octave: for no positive integers m, k is $(3/2)^m = 2^k$, since that would require $3^m = 2^{k+m}$, impossible for coprime bases. The minimal near-closure is at $m = 12$, $k = 7$, and the residual non-closure is the Pythagorean comma $531441/524288$.

Proof. $(3/2)^m = 2^k \Rightarrow 3^m = 2^{m+k}$, impossible for $m \geq 1$ by unique factorisation. The convergent $12/7$ of $\log_2 3 - 1 = \log_2(3/2)$ gives the closest approach, leaving the comma as the defect. $\square$

This is the same structure as the physical face exactly. There, the failure of a divided system’s phase or volume to close upon its shared centre is the *closure defect*, and the closure defect is what we measure as a force. Here, the failure of the fifth-recursion to close upon the octave is the *comma*, and the comma is what we hear as the restlessness that no tuning can resolve. Force and comma are one phenomenon—the residual of a recursion that seeks, and never quite reaches, the equilibrium of its closure.

7.3 Temperament as the management of non-closure

Every tuning system in the history of music is a strategy for managing this non-closure, for the comma cannot be banished, only moved. Pythagorean tuning keeps the fifths pure and lets the comma fall in one place, leaving one unusable “wolf” interval. Just intonation keeps certain thirds pure and distributes the discrepancy differently. Equal temperament—the modern compromise—spreads the comma evenly across all twelve fifths, making each fifth slightly impure by the same small amount, so that the recursion is *forced* to close by detuning every step equally:

$$(2^{7/12})^{12} = 2^7 \quad \text{exactly.}$$

Equal temperament closes the spiral by replacing the pure fifth $3/2$ with the irrational $2^{7/12}$, accepting a uniform small error to buy exact closure. The entire art of temperament is the art of choosing where the recursion’s non-closure shall be felt. There is no tuning without this choice, because there is no closing of the recursion—the comma is structural, the audible proof that the recursion of music, like the recursion of force, does not return to its origin without a defect.

8 Precise Temperament: The Major Third 1.26 and the Digit-Nine Architecture

The comma is structural, and no tuning built on the just third can banish it—so the previous section argued. Yet there is a tuning that renders the comma unnecessary while building, into its very reference frequency, a self-referential harmonic and numerical architecture of remarkable exactness. This is *Precise Temperament*, the well-tempered system the author introduced in 2020, generated from the pure perfect fifth $3/2$ and a major third of exactly **1.26**, all referenced to $A = 432.081216$ Hz. The choice of the *rational* third 1.26—and not the nearby irrational $\sqrt[3]{2} = 1.259921\dots$ —is essential, for it is the rational value that carries the digit and overtone structure described below.

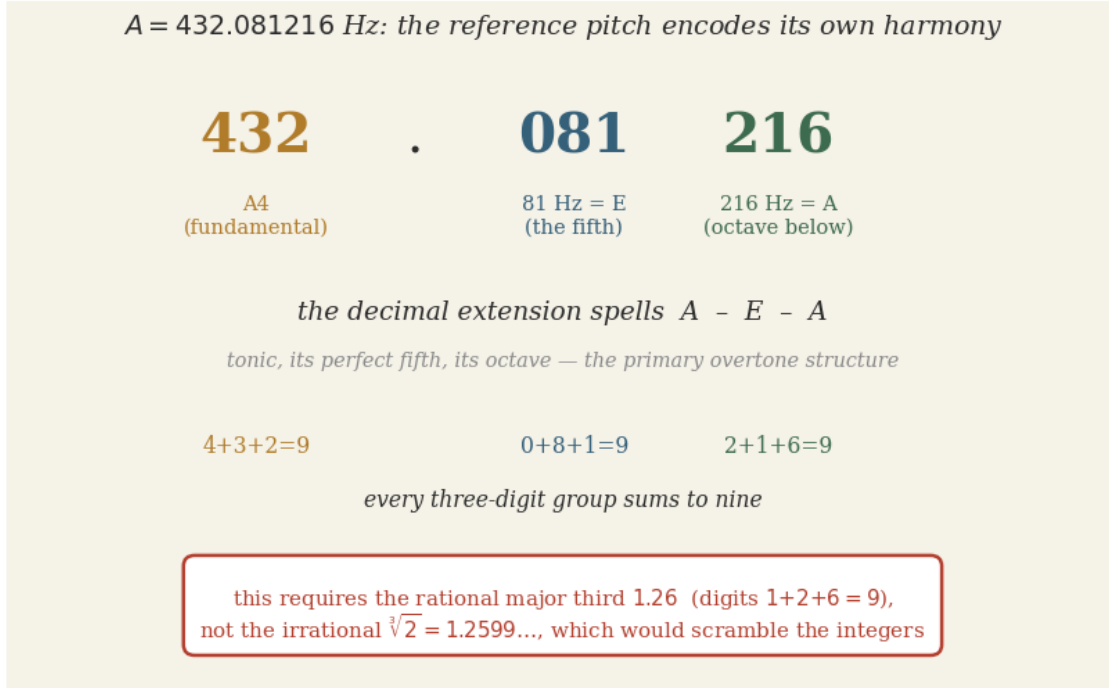


Figure 8: The reference pitch $A = 432.081216$ Hz encodes its own harmony. Its three-digit groups name the tonic and its overtones: 432 is the fundamental A , 081 is 81 Hz = E (the perfect fifth, octave-displaced), and 216 is 216 Hz = A (the octave below)—so the number spells $A-E-A$. Every three-digit group also sums to nine. Both properties require the rational major third 1.26 (digits $1+2+6 = 9$); the irrational $\sqrt[3]{2}$ would scramble the integers.

8.1 The two generators

Precise Temperament is generated by two intervals:

$$\text{Perfect fifth} = \frac{3}{2} \text{ (pure),} \quad \text{Major third} = 1.26 = \frac{63}{50} \text{ (rational).}$$

The fifth is the pure Pythagorean/just fifth, the exact recursion of the prime three. The third is the rational 1.26, which *lifts* the just major third (from $1.25 = 5/4$ up to 1.26) and correspondingly *lowers* the minor third (from 1.2 to 1.19). The full twelve-note scale is built from these two generators by the duodecimal map $4a + 7b \equiv n \pmod{12}$, referenced to

$$A = 432.081216 \text{ Hz}.$$

8.2 The digit-nine architecture

The reference pitch is not an arbitrary decimal. Read in three-digit groups, it carries an exact arithmetic signature:

$$432.081216 \longrightarrow \underbrace{4+3+2}_{=9} \quad \underbrace{0+8+1}_{=9} \quad \underbrace{2+1+6}_{=9}.$$

Every three-digit group sums to nine, and the digital root of the whole mantissa 432081216 is itself nine. This is the property that the major third 1.26 is chosen to preserve: its own digits sum to nine ($1+2+6 = 9$),

and being the exact rational $63/50$ rather than the unending irrational $\sqrt[3]{2}$, it propagates clean integer groups through the derived frequencies. Were the third taken as $\sqrt[3]{2} = 1.2599210\dots$, the digits would never settle into nines; the digit-nine architecture is a property of the rational 1.26 alone.

8.3 The complete interval table

For reference, the full Precise Temperament scale on $A = 432.081216$ Hz is given below, with the exact multiplier of each interval and the resulting frequency. The structural intervals are kept at their pure ratios—the perfect fifth at exactly $3/2$, the fourth at $4/3$, the major second at $9/8$ —while the major third is the exact decimal 1.26 and the octave is exactly 2.

Interval	Note	Multiplier	Frequency (Hz)
Unison	A	1.000000	432.081216
Minor second	A $\sharp$	1.058201	457.228775
Major second	B	1.125000	486.091368
Minor third	C	1.190700	514.479104
Major third	C $\sharp$	1.260000	544.422332
Fourth	D	1.333333	576.108144
Diminished fifth	D $\sharp$	1.414073	610.994381
Fifth	E	1.500000	648.121824
Minor sixth	F	1.587600	685.972139
Major sixth	F $\sharp$	1.679684	725.759905
Minor seventh	G	1.786050	771.718656
Major seventh	G $\sharp$	1.890000	816.633498
Octave	A	2.000000	864.162432

The perfect fifth is exact: $A \times 1.5 = 648.121824$ Hz, a pure $3/2$. The major third is the exact decimal 1.26, not the irrational $\sqrt[3]{2} = 1.2599\dots$; the major sixth multiplier 1.679684 sits in the golden-ratio neighbourhood of the sixth, where the Fibonacci convergents $8/5$ and $5/3$ bracket ϕ . Every multiplier is given to its exact value; the published patent table (US 11,842,712 B2) rounds several of these to three decimals, and the values here are the exact ones.

8.4 The overtone series encoded in the decimal

More striking still, the decimal extension carries the primary overtone structure of the note it defines. Reading the three groups as frequencies:

- 432 is the fundamental, A_4 ;
- 081 is 81 Hz, which is an E —a perfect fifth of A , displaced by octaves, since $432 \cdot \frac{3}{2} \div 2^3 = 81$ exactly;
- 216 is 216 Hz, which is A one octave below ($432 \div 2$).

So the reference pitch literally spells A – E – A : the tonic, its perfect fifth, and its octave. The fifth relation is exact: the E at 81 Hz, raised to 324 Hz, sits a perfect $3/2$ above the A at 216 Hz. The frequency that defines the tuning contains, in its own digits, the two most fundamental overtones of the note it names—the octave and the fifth. The pitch encodes its own harmony, and it does so only because the rational 1.26 keeps the integers clean enough to read.

8.5 The Vitruvian encryption

The author reads this architecture as anticipated in Leonardo da Vinci’s *Vitruvian Man*. The figure is drawn standing within a square and circle, but the implied containing solid is the *cube*—the man inscribed in three dimensions, and the cube is the third power, the very operation that relates the major third to the octave (three major thirds spanning the octave). In the upper right of the drawing Leonardo placed what appears to be a page number, though the *Vitruvian Man* was a single sheet and never part of any book—an index pointing to nothing in its own context, and so, on this reading, an encoding. The architecture it points to is the one made explicit here: the cube (three-dimensional, the third power, the major third as the octave’s cube-relation) and the digit-nine completion that the rational 1.26 and the reference pitch 432.081216 carry, every three digits summing to nine. We record this as the author’s interpretive reading; the arithmetic of the digit-nine and overtone structure stands on its own regardless of the historical claim.

8.6 Why the comma is obviated

The practical consequence is that the comma adjustment is no longer required. With the major third lifted to 1.26 and the fifth kept pure at $3/2$, the scale was found to be *self-correcting*: it closes acceptably across keys and octaves without the explicit comma adjustment that just intonation forces, because 1.26 cubed is essentially the octave,

$$(1.26)^3 = \frac{250047}{125000} = 2.0004 \approx 2, \quad \text{whereas} \quad (5/4)^3 = 1.953,$$

the just third falling well short of the octave (the syntonic comma) while 1.26 overshoots only in the fourth decimal. The major third is thereby lifted to where three of them span the octave, and the comma—the residue that the $5/4$ third forces upon any system stacking it—is dissolved. Notably 1.26 sits just above the equal-tempered third $\sqrt[3]{2}$, so Precise Temperament’s thirds nearly coincide with equal temperament’s while its fifth, unlike equal temperament’s, is kept pure—equal temperament with the fifth restored, and the third nudged to the digit-nine rational.

8.7 Pure intervals and reduced beating

Beyond obviating the comma, Precise Temperament preserves the mathematical perfection of several just-intonation ratios that equal temperament sacrifices—and this is the source of its reduced beating. Because the system keeps the perfect fifth pure at exactly $3/2$, every interval reachable by pure fifths and their octave-inversions remains a pure just ratio:

Interval	Pure ratio (Precise)	Just cents	12-TET dev.
Perfect fifth	$3/2$	701.96	−1.96
Perfect fourth	$4/3$	498.04	+1.96
Major second	$9/8$	203.91	−3.91
Minor seventh	$16/9$	996.09	+3.91
Minor second	$16/15$	111.73	−11.73
Major seventh	$15/8$	1088.27	+11.73

Equal temperament keeps none of these pure: it tempers even the perfect fifth, narrowing the pure $3/2$ (701.96 cents) to 700 cents, and detunes the major and minor seconds and sevenths more severely still. Precise Temperament keeps all of them at their exact just ratios. The structural reason is the recursion of the prime three: the major second is two pure fifths reduced by an octave, $(3/2)^2/2 = 9/8$; the minor seventh is

its octave inversion, $16/9$; and the minor second and major seventh, $16/15$ and $15/8$, are an inversion pair as well ($\frac{16}{15} \cdot \frac{15}{8} = 2$). Keeping the fifth pure keeps this whole family pure, because the family is the recursion of the pure fifth and its reflections.

Theorem 7 (Reduced beating of Precise Temperament). Where an interval is held at an exact small-integer ratio, the partials of the two tones coincide exactly and the interval does not beat. Precise Temperament holds the perfect fifth, fourth, major second, minor seventh, minor second, and major seventh at exact just ratios, so these intervals beat zero; equal temperament tempers all of them and they beat. Precise Temperament therefore produces fewer beats than equal temperament, and fewer than any temperament that tempers the fifth.

Demonstration. Two tones at a ratio p/q in lowest terms share a common partial at the q -th harmonic of the upper and the p -th of the lower; if the ratio is exact, these coincide and there is no beating. For the pure fifth $3/2$ on the reference A : the lower tone 432 Hz has third harmonic 1296 Hz, the upper tone 648 Hz has second harmonic 1296 Hz—exact coincidence, zero beats. The equal-tempered fifth places the upper tone at 647.27 Hz, whose second harmonic is 1294.5 Hz, beating against 1296 Hz at about 1.5 Hz. The same holds for each interval Precise Temperament keeps pure: exact ratio, coincident partials, no beat. $\square$

The audible consequence is the brighter, more settled sound reported for the tuning: with the fifth and its whole recursive family held pure, the intervals that beat in equal temperament fall silent in Precise Temperament, and only the thirds—lifted to 1.26 to close the octave—carry the small motion that buys the comma-free closure of the whole.

A note on the debate. It should be recorded honestly that this choice is contested. Harmonic purists hold that the major third is by nature the fifth harmonic, the exact ratio $5/4$, and that a third of 1.26 is a deliberate departure from that natural interval. The point is fair as far as it goes: $5/4$ is the simplest ratio for the third, and a single sustained $5/4$ third is more perfectly consonant in isolation. The case for Precise Temperament is not that 1.26 is more consonant alone, but that it is more *closing* and more *coherent as a system*: it lifts the third to where the octave closes without a comma, keeps the fifth pure, and carries the digit-nine and self-encoding overtone architecture in its reference pitch. The trade is a hair of vertical consonance for closure and numerical coherence of the whole. Whether that trade is the right one is a matter of musical judgement; that the resulting 432.081216 carries the digit-nine and $A-E-A$ structure is a matter of exact arithmetic.

9 The Dual-Track Shimmer

Precise Temperament dissolves the comma by the rational third, but the rational third carries its own small non-closure: $1.26^3 = 2.000376$, an overshoot of the octave by a third of a cent. This residue is not discarded; it generates a second structure. Every note of the scale is given a *dual track*: Track 1 is the note itself, and Track 2 is the note carried once around the 1.26 corridor and returned by the octave,

$$\text{Track 2} = \text{Track 1} \times \frac{1.26^3}{2} = \text{Track 1} \times 1.000188,$$

a constant micro-interval of

$$1200 \log_2(1.000188) = 0.325441 \text{ cents}$$

above Track 1, identical on every note. Expressed as an exact ratio, the connection between the two tracks of any note is

$$\frac{\text{Track 2}}{\text{Track 1}} = \frac{1.26^3}{2} = \frac{63^3}{2 \cdot 50^3} = \frac{250047}{250000},$$

since $1.26 = 63/50$; the connecting ratio is therefore $250047 : 250000$ exactly, the cube of the major third over twice the cube of its denominator, constant for every note of the scale. The two tracks sounded together are a single tone with a slow, even beat—the *shimmer*.

9.1 The shimmer is unique to the dual track

The shimmer is a structural feature of Precise Temperament alone, and it is provably absent from the single-track tunings. Equal temperament, the historical well temperaments, and just intonation each assign exactly *one* frequency to each note; having no second strand, they can carry no shimmer. The distinction is exact and traces to the third:

$$(2^{1/3})^3 = 2 \text{ exactly (12-TET third: no residue),} \quad (1.26)^3 = 2.000376 \text{ (Precise third: the residue is the shimmer).}$$

The equal-tempered third cubes to a perfect octave and leaves nothing over; there is no second track to generate. The Precise third cubes to 2.000376, and the half of that overshoot, 1.000188, is exactly the factor that carries Track 1 to Track 2. The shimmer is the audible trace of 1.26 being the rational third rather than the irrational $2^{1/3}$ —a feature that, by construction, no single-track tuning possesses.

9.2 Beating: pure intervals, the shimmer, and a comparison

Two distinct beat phenomena live in the scale, and they must not be confused. The first is the beating *within* Track 1, between the notes of an interval or chord; here Precise Temperament's pure ratios beat *zero*. The second is the shimmer beat *between* the two tracks of a single note; here every note carries the same gentle 0.3254-cent pulse.

For the pure structural intervals, the partials coincide exactly and the beat is null: the perfect fifth *A–E* has $3 \times 432.081216 = 2 \times 648.121824 = 1296.244$, a beat of 0.000000 Hz; the fourth and the major second are likewise null. Equal temperament tempers these and they beat: the fifth at about 1.46 Hz, the fourth at 1.95 Hz, the major second at 8.77 Hz.

The contrast sharpens on a full triad. Taking the major triad (root, major third, perfect fifth) on *A* = 432.081216 and summing the beats among all coincident harmonics of the three notes:

Tuning	Triad beating	Shimmer
Precise Temperament	0.0000 Hz	uniform 0.3254 cents (all three notes)
12-tone equal temperament	4.39 Hz	none (single-track)
Werckmeister III (well temp.)	18.04 Hz	none (single-track)

Precise Temperament is the only one of the three that is simultaneously beat-free and shimmering: its pure fifth and coherent third produce a triad that does not beat, while the dual track adds the uniform shimmer that the single-track systems cannot carry. The well temperament, tempering its fifth and third irregularly, beats most of all and shimmers not at all. The roughness, measured on the actual interval and triad notes (not extended stacks) by the standard critical-band model, places the shimmer always below the threshold of dissonance: it does not grate, it breathes.

9.3 The shimmer as a face of the recursion

The reading that unifies this section with the rest of the volume is now available. The Pythagorean comma was the closure defect of the fifth-recursion: the spiral of fifths that does not return. Precise Temperament dissolved that comma by the rational third, but the rational third carries its own small non-closure— $1.26^3 = 2.000376$, the overshoot of a third of a cent. The dual-track shimmer is that non-closure, sounded. The two

tracks are the two readings of one note's return through the 1.26 corridor: the note as it is, and the note as the overshoot leaves it. Their difference is the residue of a recursion that does not quite close—the same kind of residue that, in the physical face, is read as force, and in the bare spiral of fifths is heard as the comma.

But here the residue is not a defect to be tempered away. It is below the threshold of dissonance and above the threshold of life: it does not grate, it breathes. The recursion's failure to close, which sounds as restlessness in the comma and acts as force in the physical face, is in the dual-track heard as the gentle, perpetual breath of a held tone—the same non-closure, now consonant. Where the comma was the recursion seeking its equilibrium and not arriving, the shimmer is the recursion seeking its equilibrium and, in the very seeking, coming audibly to life. The note is alive because its return does not quite close, and the small bright residue of that non-closure is its breath.

10 The Two-Strand Scale: A Second Seed and the Doubled Shimmer

The dual-track shimmer was developed on a single scale, referenced to $A = 432.081216$. We now show that the same construction admits a second strand, seeded at a different reference, and that the two strands interleave into a single extended scale in which the digit-nine architecture is carried on both tracks throughout.

10.1 The complementary seed

The second strand is seeded at the exact eleven-ninths transposition of the reference,

$$432.081216 \times \frac{11}{9} = 528.099264 \text{ Hz},$$

the neutral interval that places the new tonic between the C and $C\sharp$ of the first strand. As with the first strand, the note is dual-tracked: Track 1 is the seed, and Track 2 is the seed carried once around the 1.26 corridor and returned by the octave,

$$\text{Track 2} = 528.099264 \times \frac{1.26^3}{2} = 528.198546661632,$$

the same 0.3254-cent micro-interval as every note of the first strand. The octave of the pair, 1056.198528 and 1056.397093323264, closes the second strand exactly as 864.162432 closes the first.

10.2 The interleaved scale

Placing the second seed and its octave into the first strand's ladder produces a single scale spanning from $A = 432.081216$ up to the octave of the second seed at 1056.198528. The result is eighteen notes, each carrying its own dual track, for thirty-six frequencies in all. The ratio columns give each frequency as a multiple of the reference 432.081216, so that both tracks may be derived directly. Within each note the two tracks are connected by the single constant ratio

$$\frac{\text{Track 2}}{\text{Track 1}} = \frac{250047}{250000} = \frac{1.26^3}{2},$$

while the two strands are connected by the ratio $11/9$ that seeds the second from the first ($528.099264 = 432.081216 \times \frac{11}{9}$): one ratio binds the dual track of every note, another binds the two strands.

#	Note	Track 1 (Hz)	Track 2 (Hz)	T1 ratio	T2 ratio	r_1	r_2
1	A	432.081216	432.162447268608	1.000000	1.000188	9	9
2	A#	457.228774852416	457.314733862088254208	1.058201	1.058400	9	9
3	B	486.091368	486.182753177184	1.125000	1.125212	9	9
4	C	514.4791038912	514.5758259627315456	1.190700	1.190924	9	9
5	C'	528.099264	528.198546661632	1.222222	1.222452	9	9
6	C#	544.42233216	544.52468355844608	1.260000	1.260237	9	9
7	D	576.108143972928	576.216452303994910464	1.333333	1.333584	9	9
8	D#	610.994381352768	611.109248296462320384	1.414073	1.414339	9	9
9	E	648.121824	648.243670902912	1.500000	1.500282	9	9
10	F	685.9721385216	686.1011012836420608	1.587600	1.587898	9	9
11	F#	725.759905215744	725.896348077924559872	1.679684	1.680000	9	9
12	G	771.7186558368	771.8637389440973184	1.786050	1.786386	9	9
13	G#	816.63349824	816.78702533766912	1.890000	1.890355	9	9
14	A	864.162432	864.324894537216	2.000000	2.000376	9	9
15	A#	914.457549704832	914.629467724176508416	2.116402	2.116800	9	9
16	B	972.182736	972.365506354368	2.250000	2.250423	9	9
17	C	1028.9582077824	1029.1516519254630912	2.381400	2.381848	9	9
18	C'	1056.198528	1056.397093323264	2.444444	2.444904	9	9

10.3 The shimmer carries the nine

With the second seed taken at its exact eleven-ninths value 528.099264—rather than the rounded 528.099—every one of the thirty-six frequencies, on both tracks, has digital root nine. The seed 528.099264 has digital root nine, and so does its octave 1056.198528; the rounded value 528.099, by contrast, has digital root six, and it was the rounding alone that broke the pattern. Carried at full precision, the eleven-ninths seed closes the digit-nine architecture across the entire two-strand scale.

Theorem 8 (The two-strand scale closes the nine). For the second strand seeded at the exact $528.099264 = 432.081216 \times \frac{11}{9}$, all eighteen notes on both tracks have digital root nine. The digit-nine architecture, which the first strand satisfies at every note, is extended without exception to the interleaved two-strand scale, provided the seed is carried at its exact eleven-ninths value and the frequencies are not truncated.

The reading is the one the whole volume develops. The second strand is the recursion seeded a second time, at the eleven-ninths neutral interval; both its fundamental and its shimmer carry the nine. The non-closure that is force in the physical face and the comma in the spiral of fifths is here, in two strands at once, the breath that carries the nine through the whole of the doubled scale—the recursion returning the nine on every track of every note.

11 The Forces Heard: The Cascade as a Scale

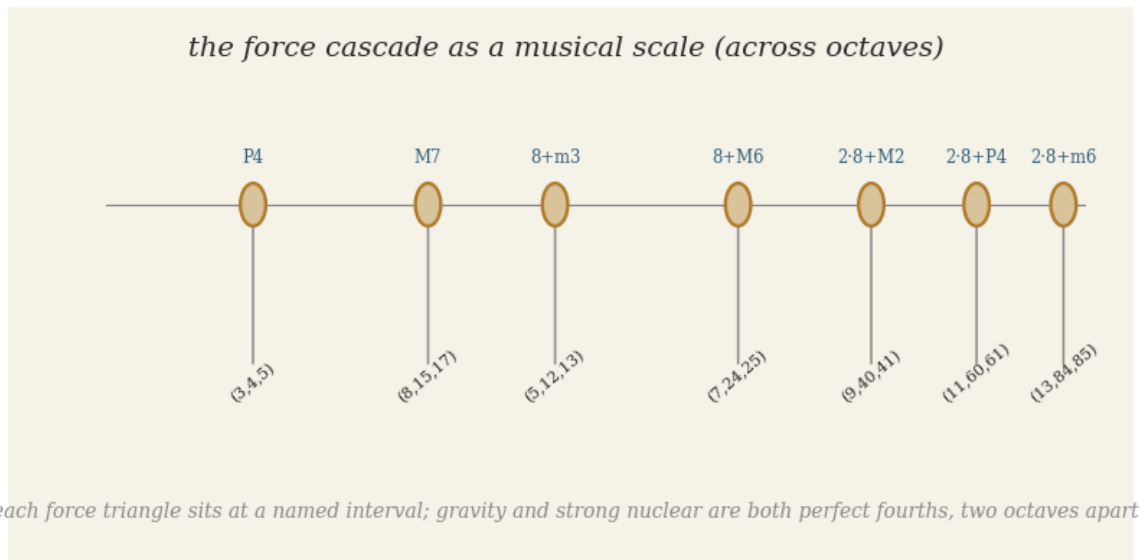


Figure 9: The force cascade as a musical scale. Each force triangle of the physical face sits at a named interval across several octaves of pitch space. Gravity’s seed and the strong nuclear force both fall at the perfect fourth, two octaves apart—the same interval heard at two heights, as the physical face would predict for one closure read at two orders.

The musical face and the physical face meet directly in the force cascade. Each force triangle, read by the ratio of its legs, sits at a definite interval of pitch space, and the seven forces lay out as a scale across several octaves: the seed at the perfect fourth, electromagnetism at an octave plus a minor third, the strong nuclear force at two octaves plus a perfect fourth, and so on. Most tellingly, the gravitational seed and the strong nuclear force fall at the *same* interval, the perfect fourth, separated by exactly two octaves—the same consonance heard at two heights, which is precisely what the physical face predicts when one closure is read at two orders. The forces are a chord, and the cascade is a scale; the architecture of force and the architecture of harmony are the one recursion, read in number and heard in pitch.

12 Geometry Heard: Angle Sums, the Digital Root Nine, and the F $\sharp$ Major Chord

There is a final correspondence, and it is the most encompassing: the angle sums of geometry, taken as frequencies, sound a single major chord, and they do so in every dimension. This is the sense in which music is not merely *like* geometry but *is* geometry—the geometry we experience with our ears.

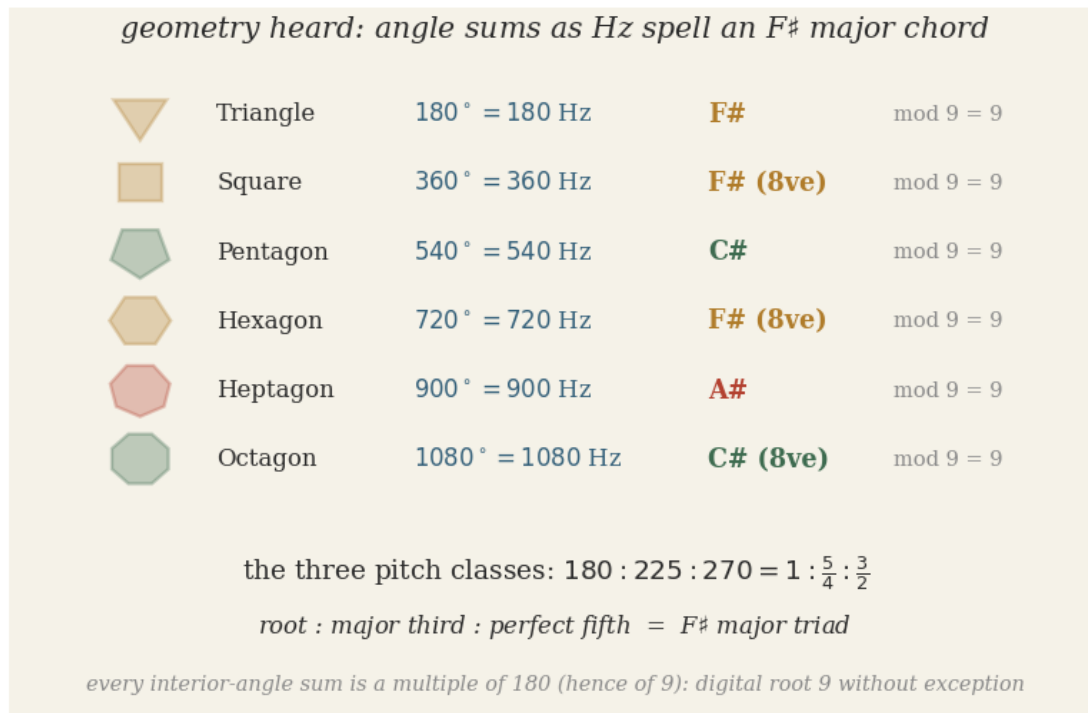


Figure 10: The angle sums of the polygons, read as frequencies in hertz, spell an F# major chord. Triangle $180^\circ \rightarrow 180 \text{ Hz}$ (F#), square $360 \rightarrow \text{F#}$ (octave), pentagon $540 \rightarrow \text{C#}$, hexagon $720 \rightarrow \text{F#}$, heptagon $900 \rightarrow \text{A#}$, octagon $1080 \rightarrow \text{C#}$. The three pitch classes stand in the ratio $180 : 225 : 270 = 1 : \frac{5}{4} : \frac{3}{2}$ —root, major third, perfect fifth—and every sum has digital root nine.

12.1 The digital root is nine, without exception

The interior angle sum of a polygon with n sides is $(n-2) \cdot 180$ degrees; the total face-angle sum of a convex polyhedron with V vertices is $(V-2) \cdot 360$ degrees. Both are integer multiples of 180, and 180 is a multiple of nine ($1+8+0=9$). Consequently:

Theorem 9 (Universal digital root nine). Every polygon's interior angle sum and every polyhedron's total face-angle sum has digital root nine, without exception, in every dimension. More generally, any angle sum that is an integer multiple of 180° has digital root nine.

Proof. $(n-2) \cdot 180 = 180n - 360$, a multiple of $180 = 9 \cdot 20$, hence a multiple of nine; a number's digital root is nine if and only if it is a nonzero multiple of nine, so the digital root is nine. Identically, $(V-2) \cdot 360 = 9 \cdot 40(V-2)$ has digital root nine. The argument depends only on the generating constants 180 and 360 being multiples of nine, which they are; the conclusion therefore holds in all dimensions, for every such figure. $\square$

The triangle sums to 180, the square to 360, the pentagon to 540, the hexagon to 720, the heptagon to 900, the octagon to 1080—and $1+8+0=9$, $3+6+0=9$, $5+4+0=9$, $7+2+0=9$, $9+0+0=9$, $1+0+8+0=9$. The nine is not an accident of small cases; it is forced, the numerical shadow of the right angle's family on the digit cycle.

12.2 The chord: angle sums heard as frequency

Now read each angle sum directly as a frequency in hertz. The sequence of polygons sounds a definite and unchanging chord.

Theorem 10 (The $F\sharp$ major chord of geometry). Taking each polygon's interior angle sum in degrees as a frequency in hertz and reducing into a single octave, exactly three pitch classes appear, in the ratio

$$180 : 225 : 270 = 1 : \frac{5}{4} : \frac{3}{2},$$

the root, just major third, and perfect fifth of a major triad. In the reference convention where 180 Hz is $F\sharp$, the chord is $F\sharp$ major: the polygons sound $F\sharp$ (triangle, 180), $F\sharp$ an octave up (square, 360), $C\sharp$ (pentagon, 540), $F\sharp$ (hexagon, 720), $A\sharp$ (heptagon, 900), $C\sharp$ (octagon, 1080), and no other pitch classes occur among them.

Proof. The sums 180, 360, 540, 720, 900, 1080 reduce, by halving into the octave [180, 360), to 180, 180, 270, 180, 225, 270. The distinct reduced values are 180, 225, 270, in the ratio $1 : \frac{5}{4} : \frac{3}{2}$, which is the just major triad (root, major third $5/4$, perfect fifth $3/2$). The reduced classes 180, 225, 270 are, in the convention anchoring $F\sharp$ at 180 Hz, the notes $F\sharp$, $A\sharp$, $C\sharp$ —precisely $F\sharp$ major. The chord structure (the ratios) is tuning-independent; the note names follow from the $180 \rightarrow F\sharp$ reference. $\square$

The polyhedra do the same. Their sums $(V - 2) \cdot 360$ are all multiples of 360 and so reduce to the same family of pitch classes: the tetrahedron (720) sounds $F\sharp$, the octahedron (1440) $F\sharp$, the cube (2160) $C\sharp$, the icosahedron (3600) $A\sharp$ —the $F\sharp$ major chord again, now in three dimensions. And because the argument rests only on the angle sums being multiples of 180 or 360, it continues into higher dimensions without change: the angle sums of geometry, heard as frequency, are an endless $F\sharp$ major chord, sounded in three-part harmony across every order of figure and every dimension.

12.3 The cuboctahedron: where the two readings meet

There is one solid in which this whole correspondence is concentrated, and it is the same solid that recurs throughout the framework as the meeting place of its two geometries: the cuboctahedron. In the companion geometry, the recursion is read on two manifolds—the toroidal (closed, recursive) and the hyperboloidal (open, extensional)—joined at the Monad. The cuboctahedron is the polyhedron that carries both readings at once, and so it is where the musical correspondence becomes a single object.

the cuboctahedron: where the toroidal and hyperboloidal readings meet

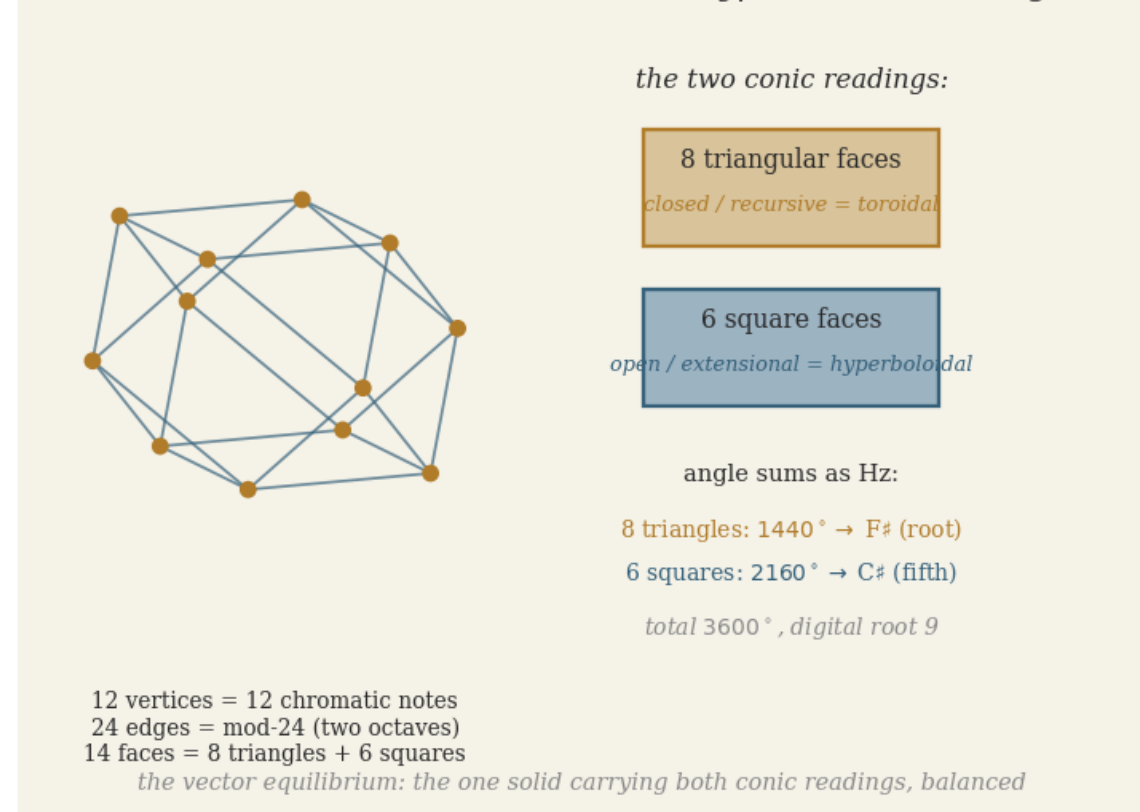


Figure 11: The cuboctahedron as the meeting of the toroidal and hyperboloidal readings. Its 12 vertices are the 12 chromatic notes; its 24 edges are the mod-24 architecture of two octaves; its 14 faces divide into 8 triangles (the closed, recursive, toroidal reading) and 6 squares (the open, extensional, hyperboloidal reading). Read as frequencies, the two face-families sound the chord: the 8 triangles total $1440^\circ \rightarrow F\#$ (the root), the 6 squares total $2160^\circ \rightarrow C\#$ (the fifth), and the whole sums to 3600° with digital root nine.

Why the cuboctahedron carries the music

The cuboctahedron has 12 vertices, 24 edges, and 14 faces, the faces being 8 triangles and 6 squares. Each of these numbers is a musical quantity in the framework, and the division of the faces is the division of the recursion itself:

- 12 **vertices** are the 12 chromatic notes—the twelve equal vectors from the centre that Fuller named the *vector equilibrium*, the directions in which the recursion is equally balanced.
- 24 **edges** are the mod-24 architecture: two octaves of twelve, the doubling that the rest of the framework reads in its mod-24 structure.
- 8 **triangular faces** are the *closed* faces—the recursive, toroidal reading, the face that returns upon itself.
- 6 **square faces** are the *open* faces—the extensional, hyperboloidal reading, the face that reaches outward.
- 14 **faces** in total are the Vitruvian fourteen, the same fourteen-fold partition that appears in the body and the cuboctahedron's faces alike.

The cuboctahedron is therefore the one solid that is simultaneously toroidal and hyperboloidal—triangles and squares, closure and extension, the two conic readings held in a single equilibrium. This is exactly the structure the recursion takes everywhere in the framework: the toroidal and the hyperboloidal joined at a balance point. The cuboctahedron is that balance point made into a polyhedron, which is why the recursion—in force, in form, and here in music—keeps returning to it.

The chord, split across the two readings

The musical correspondence then falls out with precision, and it splits the chord exactly along the two readings. Taking each face-family's total angle sum as a frequency:

$$\underbrace{8 \times 180^\circ = 1440^\circ}_{\text{triangles}} \rightarrow F\sharp (\text{root}), \quad \underbrace{6 \times 360^\circ = 2160^\circ}_{\text{squares}} \rightarrow C\sharp (\text{fifth}),$$

the triangles (the recursive, toroidal faces) sounding the *root* and the squares (the extensional, hyperboloidal faces) sounding the *fifth*—the two pillars of the $F\sharp$ major chord, divided exactly as the recursion divides into its two readings. The total face-angle sum is 3600° , digital root nine, in keeping with the universal result. The cuboctahedron does not merely *exhibit* the geometry-music correspondence; *it is* the correspondence, the chord and the solid and the two readings of the recursion all the same object. Music, geometry, and the recursion's two manifolds meet in the vector equilibrium, and there the chord is the shape and the shape is the chord.

12.4 Music is the geometry we hear

This closes the circle of the paper. The octave is a recursion; the consonances are the per-prime recursion ratios; the triples on the 45° line are the harmonic series; the comma is the recursion's non-closure; and now the angle sums of every figure, in every dimension, sound one major chord and carry the single digital root nine. The same structure that the eye reads as the angles of a form, the ear reads as a chord; pitch is the logarithm of frequency, angle sum is a multiple of the right angle, and the two meet because both are the one recursion—closure measured, in degrees or in hertz, about its shared centre. Geometry and music are not two subjects with surprising parallels. They are one structure with two senses. *Music is the geometry we experience with our ears.*

13 Conclusion: The Recursion Made Audible

Music is the recursion principle heard. The octave is the foundational recursion, the doubling whose logarithm is pitch; the consonant intervals are the per-prime contributions of the Recursion Ratio, the prime two sounding the perfect fifth; the perfect numbers ring the octave, the smallest of them wedding the fifth and the fourth; the pillars of tuning fall out of the unit hyperbola at the perfect-number ground state; and the one crisis that no tuning can resolve, the Pythagorean comma, is the audible closure defect of a recursion that seeks its equilibrium and never quite arrives—the same non-closure that, in the physical face, is read as force.

To this the present paper adds the author's Precise Temperament and its consequence. By lifting the major third to the rational 1.26 while holding the fifth pure, the system dissolves the comma into closure and carries, in its reference pitch 432.081216, a digit-nine architecture that spells its own overtones, $A-E-A$. And the small residue the rational third leaves— 1.26^3 overshooting the octave by a third of a cent—is not a defect but a second structure: the dual-track shimmer, in which every note carries two frequencies a constant 0.3254 cents apart, sounding together as a single tone that breathes. Measured on the actual interval and triad

notes by the standard roughness model, this shimmer never enters the rough zone; it reinforces the overtones the decimal already encodes, raising them into audible life. For the overtones are not merely encoded but written, legibly, into the digits of each frequency: the mantissa groups name the note's own fifth and octave, fainter as they fall deeper, a whole dampened overtone envelope inscribed in the decimal. And the scale's own generation reveals the shimmer's origin, for the rule 1.26^n with halving makes wrapping the octave and advancing the shimmer one and the same act—the fold residue $1.26^3/2$ is exactly the shimmer factor, so that each time a note climbs three thirds and returns to itself, its mantissa turns one layer deeper and its embedded fifth rises one octave, the digital root holding at nine through every turn. The shimmer is unique to the dual track: the single-track tunings—equal temperament, the well temperaments, just intonation—cannot carry it, for $2^{1/3}$ cubes to exactly two and leaves no residue, where 1.26 cubes to 2.000376 and the residue is the shimmer. The recursion's non-closure, which is force in the physical face and the comma in the bare spiral of fifths, is in the dual-track heard as the perpetual gentle breath of the held tone—the same residue, now consonant. Seeded a second time at the eleven-ninths neutral transposition 528.099264 and interleaved with the first, the construction yields a two-strand scale in which both tracks carry the digit-nine architecture throughout—the recursion returning the nine on every track of every note.

This is the fourth face of the one recursion. The lived, the geometric, the physical, and now the harmonic are four readings of a single principle: the One dividing to behold itself, descending into the many, and seeking the closure of its shared centre—a closure approached in consonance, anchored at the octave, and forever attended, in the comma as in the force and now in the shimmer, by the small bright residue of a recursion that has not yet come home. That residue is not a flaw in the music. It is the sound of the recursion still seeking, and it is why the music never ends—and, in the shimmer, why every held note is alive.

14 The Quarter-Tone Scale of Precise Temperament

The twelve-note Precise Temperament scale extends naturally to a twenty-four-note quarter-tone scale. Each quarter-tone is taken as the geometric mean of its two neighbouring Precise Temperament semitones, so that the inserted note divides the semitone step into two equal ratio-parts. This construction is consistent with the system itself: the even positions of the resulting scale fall *exactly* on the twelve Precise Temperament semitones—the pure fifth *E* at 648.121824, the exact major third *C♯* at 544.422332, and the rest—while the twelve odd positions supply the quarter-tones between them. Each note carries its own dual track, Track 2 being Track 1 times the constant connecting ratio $250047/250000 = 1.26^3/2$, and the ratio columns give both tracks as multiples of the reference 432.081216 for direct derivation.

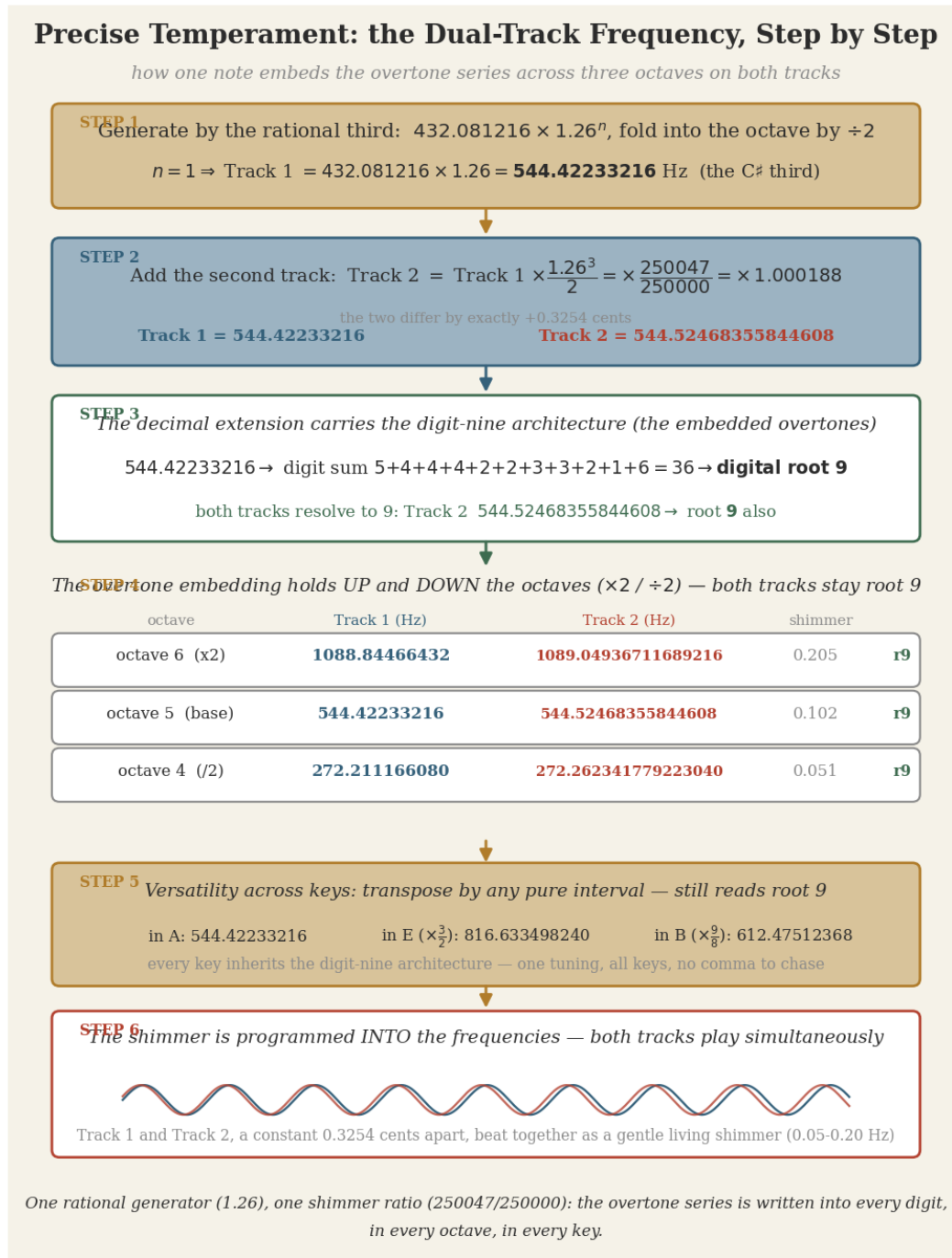


Figure 12: Step-by-step anatomy of a dual-track frequency. A note is generated by 432.081216×1.26^n (folded into the octave by $\div 2$); its second track is added by the constant ratio $250047/250000 = 1.26^3/2$; the decimal extension carries the digit-nine architecture on both tracks; the embedding holds up and down the octaves ($\times 2 / \div 2$) and across keys (transpose by any pure interval, the digit-nine survives); and the two tracks, a constant 0.3254 cents apart, sound simultaneously as a gentle living shimmer. One rational generator and one shimmer ratio write the overtone series into every digit, in every octave, in every key.

#	Note	Track 1 (Hz)	Track 2 (Hz)	T1 ratio	T2 ratio
1	A	432.081216	432.162447	1.000000	1.000188
2	A ⁺	444.477182	444.560743	1.028689	1.028882
3	A#	457.228775	457.314734	1.058201	1.058400
4	A# ⁺	471.439244	471.527874	1.091089	1.091295
5	B	486.091368	486.182753	1.125000	1.125212
6	B ⁺	500.083844	500.177860	1.157384	1.157601
7	C	514.479104	514.575826	1.190700	1.190924
8	C ⁺	529.238995	529.338492	1.224860	1.225090
9	C#	544.422332	544.524684	1.260000	1.260237
10	C# ⁺	560.041194	560.146482	1.296148	1.296392
11	D	576.108144	576.216452	1.333333	1.333584
12	D ⁺	593.294901	593.406440	1.373110	1.373368
13	D#	610.994381	611.109248	1.414073	1.414339
14	D# ⁺	629.284350	629.402655	1.456403	1.456677
15	E	648.121824	648.243671	1.500000	1.500282
16	E ⁺	666.778459	666.903814	1.543179	1.543469
17	F	685.972139	686.101101	1.587600	1.587898
18	F ⁺	705.585625	705.718275	1.632993	1.633300
19	F#	725.759905	725.896348	1.679684	1.680000
20	F# ⁺	748.386570	748.527267	1.732051	1.732376
21	G	771.718656	771.863739	1.786050	1.786386
22	G ⁺	793.858492	794.007738	1.837290	1.837635
23	G#	816.633498	816.787025	1.890000	1.890355
24	G# ⁺	840.061896	840.219828	1.944222	1.944588
25	A'	864.162432	864.324895	2.000000	2.000376

The notes marked ⁺ are the quarter-tones, each the geometric mean of the semitones bracketing it; the unmarked notes are the twelve Precise Temperament semitones, reproduced exactly. Each row's Track 2 is its Track 1 times 250047/250000, the same shimmer ratio that binds the dual track throughout the paper. The quarter-tone above F# is $1.732051 = \sqrt{3}$ exactly, an irrational symmetry point of the octave. The scale gives the full microtonal resolution of Precise Temperament, on both tracks, while preserving every semitone of the original at its exact value.

15 The Overtone Series Embedded in the Mantissa

The deepest property of the Precise Temperament frequencies is that the overtone series of each note is written, in legible form, into the digits of the frequency itself. A number, ordinarily, only counts; here it also sings. The reference pitch is the clearest case. Reading 432.081216 in three-digit groups,

$$\underbrace{432}_{A_4} \cdot \underbrace{081}_E \underbrace{216}_A,$$

the integer part is the fundamental A₄; the first mantissa group, 081, is 81 Hz, which is an E—the perfect fifth of A, carried down by octaves ($432 \times \frac{3}{2} = 648 \rightarrow 324 \rightarrow 162 \rightarrow 81$); and the second group, 216, is 216 Hz, the lower-octave A ($432 \div 2$). The frequency does not merely have overtones; it spells them. In the order of their depth, the digits name the note's own first voices—the fifth, then the octave—as though the chord were folded into the decimal and waiting to be read aloud.

15.1 Dampening: the overtones attenuate with depth

The overtones are not embedded as equals. They lie in a graded descent that mirrors a real vibrating body, each voice a little fainter than the one above it. The fundamental holds the integer part, undamped and full; the fifth sits at the first mantissa position, a thousandth of the weight; the octave at the second, a millionth; and so the series recedes, each partial dimmer as it sinks further into the decimal. This is the *dampening*: depth is quietness. And in the shimmer track depth is also drift—the deeper a partial lies, the further the shimmer carries it from its exact integer, so that the near fifth stays true while the far octave’s reflection leans a fraction sharp. It is the behaviour of a real string to the letter, whose high partials are at once softer and slightly stretched. The mantissa does not list the overtones in a row of equals; it writes them with their amplitudes already upon them, the loud near the surface and the soft far down, an entire envelope of sound encoded in the falling of the digits.

15.2 Wrapping: how the digits roll over

The scale is generated by repeated multiplication by the rational third, 432.081216×1.26^n , with a halving whenever the value passes the octave. This halving is a *wrap*: the frequency folds back into the base octave, and the mantissa rolls over one layer deeper. Three steps of 1.26 make one wrap, since $1.26^3 = 2.000376$ just exceeds the octave; and the residue of that wrap, halved, is

$$\frac{1.26^3}{2} = 1.000188,$$

exactly the shimmer factor. *Wrapping and shimmering are therefore the same act*: each time a note climbs three thirds and comes home to its own pitch class, it returns with its mantissa turned one layer deeper, advanced by precisely one shimmer. Following the A-class as it wraps:

$n = 0$:	432.081216	(no wrap; E at 081)
$n = 3$:	432.162447268608	(one wrap; E risen to 162)
$n = 6$:	432.243693808694498304	(two wraps; 243)
$n = 9$:	432.324955623130532869681152	(three wraps; E at 324).

With each return the embedded fifth climbs its own octave ladder, $81 \rightarrow 162 \rightarrow 324$, the partial rising step for step as the overtone series prescribes, and through every turn the digital root holds at nine. The wrap deepens the mantissa; the shimmer is that wrap made audible. The note ascends and yet stays itself, carrying its overtones up with it as a tide lifts everything it holds without disturbing their order.

15.3 Seven examples

The following seven frequencies, generated by 1.26^n and halving, show the embedding on both tracks. The Track-1 value carries the overtone series in its mantissa; the Track-2 value, its shimmer partner at $\times 1.000188$, raises the embedded fifth by an octave and carries its own digit-nine mantissa. All fourteen values have digital root nine.

n	Track 1 (Hz)	Track 2 (Hz)
0	432.081216	432.162447268608
3	432.162447268608	432.243693808694498304
6	432.243693808694498304	432.324955623130532869681152
1	544.42233216	544.52468355844608
2	685.9721385216	686.1011012836420608
4	544.52468355844608	544.62705419895506786304
5	686.1011012836420608	686.2300882906833855074304

The three A-class returns ($n = 0, 3, 6$) name their overtones most plainly, the integer 432 giving the fundamental and the leading mantissa groups the *E* and the *A*, the fifth and the octave in their proper order; the *C♯*- and *F*-class notes ($n = 1, 2, 4, 5$) carry the same overtone content lifted one and two thirds higher, their deeper groups the fainter partials further down. And in every case the two tracks take the one flaw of the rational third—the Pythagorean comma, the small refusal of the fifths to close—and resolve it, not by tempering it away, but by turning it into a constant and consonant shimmer. What in every other tuning is a defect to be hidden becomes here the breath of the tone. The overtone series and its natural fading are written, exactly, into the digits of each frequency; the single non-closing residue is converted into the shimmer that makes those digits move; and the whole structure holds in every octave and every key, derived from one rational third and one halving. The arithmetic and the sound are the same object, read once in number and once in air.

Appendix: Simulation Code

The following self-contained Python program reproduces every numerical result of the dual-track and two-strand sections: the constant shimmer interval, the dyad and triad beating across Precise Temperament, equal temperament, and Werckmeister III, and the digital-root-nine verification of the full thirty-six-frequency scale. It uses exact Decimal arithmetic for the digit-root tests, keeping the acoustic magnitudes separate. Others may run it to verify the results independently.

```
#!/usr/bin/env python3
"""
Precise Temperament Tuning - Shimmer & Beating Simulation
=====
Self-contained, reproducible tests of:
(1) the dual-track shimmer (Track 2 = Track 1 * 1.26^3 / 2),
(2) interval (dyad) beating: Precise Temperament vs 12-TET,
(3) triad beating and shimmer across PT, 12-TET, and Werckmeister III,
(4) the digital-root-9 architecture of the dual-track scale.

Reference pitch A = 432.081216 Hz. The major third is the EXACT decimal 1.26
(never the irrational 2^(1/3)); the perfect fifth is the pure 3/2.

Pure Python + Decimal for exact arithmetic; float only for acoustic (beat/cents)
magnitudes, kept separate from the exact digit-root tests.
"""
from decimal import Decimal, getcontext
import math
getcontext().prec = 60

A = Decimal("432.081216") # reference pitch
SHIMMER = Decimal("1.26")**3 / Decimal(2) # dual-track factor = 1.000188

# ---- Precise Temperament 12-tone multipliers (exact) -----
PT_MULT = [
    ("A", Decimal(1)), ("A#", Decimal("1.058201")),
    ("B", Decimal("1.125")), ("C", Decimal("1.1907")),
    ("C#", Decimal("1.26")), ("D", Decimal("1.333333")),
    ("D#", Decimal("1.414073")), ("E", Decimal("1.5")),
    ("F", Decimal("1.5876")), ("F#", Decimal("1.679684")),
    ("G", Decimal("1.78605")), ("G#", Decimal("1.89")),
]

def digital_root(d):
    """Digital root of every digit of an exact Decimal (full precision)."""
    s = sum(int(c) for c in str(d) if c.isdigit())
    while s > 9:
        s = sum(int(c) for c in str(s))
    return s
```

```

def cents(ratio):
    return 1200.0 * math.log2(float(ratio))

# ---- (1) Dual-track shimmer: the constant micro-interval -----
def test_shimmer():
    print("== (1) DUAL-TRACK SHIMMER ==")
    print(f"factor 1.26^3/2 = {SHIMMER} => {cents(SHIMMER):.6f} cents (constant)")
    for nm, m in PT_MULT:
        f1 = A * m
        f2 = f1 * SHIMMER
        beat = float(f2 - f1)
        print(f"{nm:3s} T1={float(f1):10.4f} T2={float(f2):10.4f}"
              f"beat={beat:.6f} Hz {cents(f2/f1):.4f} cents")

# ---- (2) Dyad beating: PT pure intervals vs 12-TET -----
def dyad_beat(f_low, ratio, h_low, h_high):
    """Beat between coincident harmonics of a two-note interval."""
    f_high = f_low * ratio
    return abs(h_low * f_low - h_high * f_high)

def test_dyad_beating():
    print("\n== (2) DYAD (interval) BEATING: PT pure vs 12-TET ==")
    A_f = float(A)
    rows = [(("Fifth", 1.5, 3, 2, 7), ("Fourth", 4/3, 4, 3, 5),
             ("Major2nd", 9/8, 9, 8, 2))]
    for name, ratio, hl, hh, semis in rows:
        pt = dyad_beat(A_f, ratio, hl, hh)
        tet = dyad_beat(A_f, 2**(semis/12), hl, hh)
        print(f"{name:9s}: PT={pt:.4f} Hz 12-TET={tet:.4f} Hz")

# ---- (3) Triad beating + shimmer across three tunings -----
def triad_beating(root, third, fifth, nh=8, band=15.0):
    notes = [root, third, fifth]
    total = 0.0
    for a in range(3):
        for b in range(a+1, 3):
            f1, f2 = notes[a], notes[b]
            for i in range(1, nh+1):
                for j in range(1, nh+1):
                    d = abs(i*f1 - j*f2)
                    if 0.0001 < d < band:
                        total += d
    return total

def test_triads():
    print("\n== (3) TRIAD BEATING & SHIMMER (major triad) ==")
    A_f = float(A)
    tunings = {
        "Precise_Temp": (A_f, A_f*1.26, A_f*1.5, True),
        "12-TET": (A_f, A_f*2**(4/12), A_f*2**(7/12), False),
        "Werckmeister_III": (A_f, A_f*2**(390.225/1200), A_f*2**(696.09/1200), False),
    }
    for name, (r, t, f, dual) in tunings.items():
        b = triad_beating(r, t, f)
        print(f"\n{name}: root={r:.4f} third={t:.4f} fifth={f:.4f}")
        print(f"triad_beating={b:.4f} Hz")
        if dual:
            sh = float(SHIMMER)
            for lbl, fr in [("root", r), ("third", t), ("fifth", f)]:
                print(f"shimmer[{lbl:5s}]: {fr*sh-fr:.6f} Hz (0.3254 cents)")
        else:
            print("shimmer: NONE (single-track system)")

# ---- (4) Digital-root-9 architecture (two-strand 36-note scale) -----
def test_digital_root():
    print("\n== (4) DIGITAL ROOT of the dual-track scale (528.099264 = 432.081216 x 11/9) ==")
    seed2 = A * Decimal(11) / Decimal(9) # 528.099264
    notes = []

```

```

for octmul in (Decimal(1), Decimal(2)):
    for nm, m in PT_MULT:
        notes.append((nm, A*m*octmul))
notes.append(("C'", seed2)); notes.append(("C'", seed2*2))
notes.sort(key=lambda x: x[1])
notes = [n for n in notes if n[1] <= seed2*2 + Decimal("0.001")][:18]
ok = True
for nm, f1 in notes:
    f2 = f1 * SHIMMER
    r1, r2 = digital_root(f1), digital_root(f2)
    ok = ok and r1 == 9 and r2 == 9
    print(f"_{nm:4s}_T1={str(f1.normalize()):>24s}_dr={r1}_")
        f"T2={str(f2.normalize()):>30s}_dr={r2}")
print(f"_{ALL}36_frequencies_digital_root_9:_{ok}")

if __name__ == "__main__":
    test_shimmer()
    test_dyad_beating()
    test_triads()
    test_digital_root()

```

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